Introduction to Magnetohydrodynamics (MHD)

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2021 SULI Introductory Course
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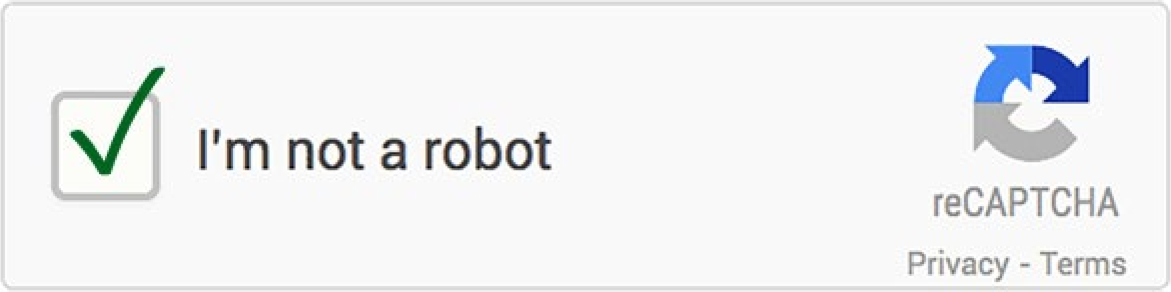
PRINCETON PLASMA PHYSICS LABORATORY



From Melbourne, Australia:

Large dog enthusiast.

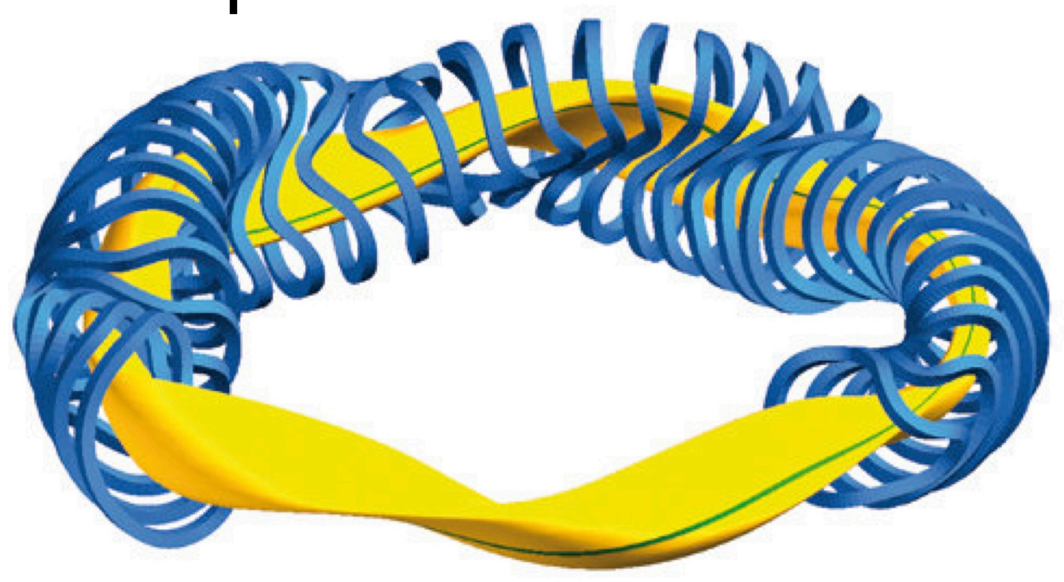
Currently here

Canberra, Australia:
BSc, PhD

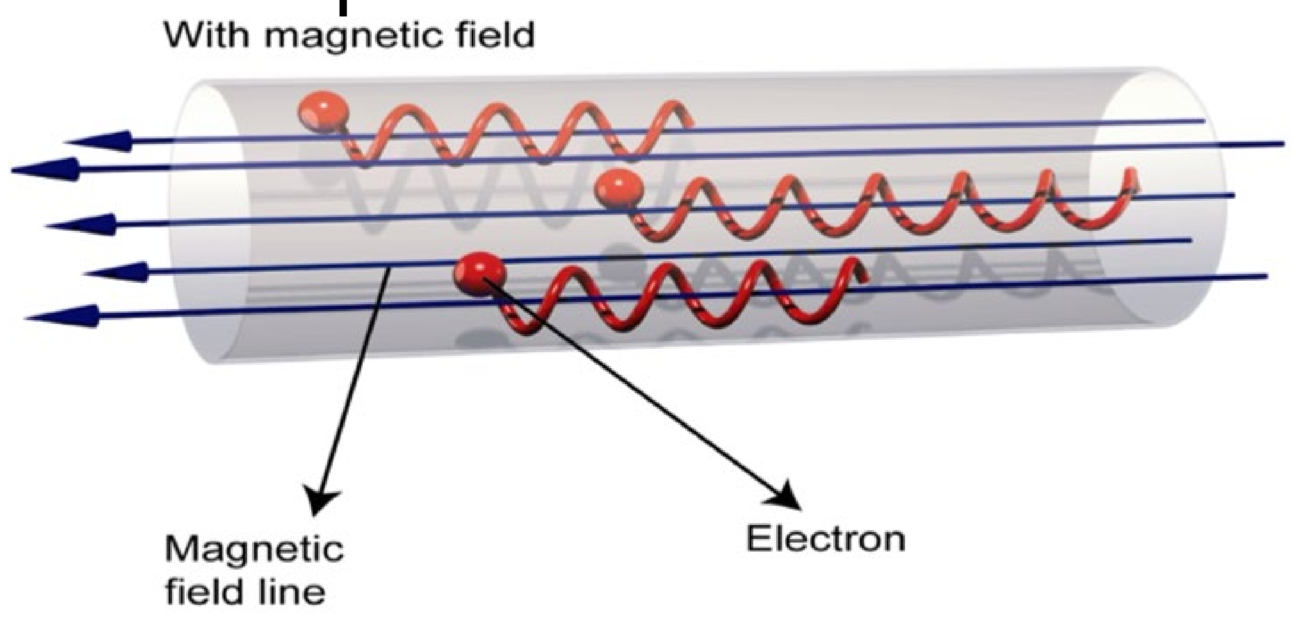
Magnetohydrodynamic (MHD) models in context

Modelling hierarchies in plasma physics

Macroscopic



Microscopic



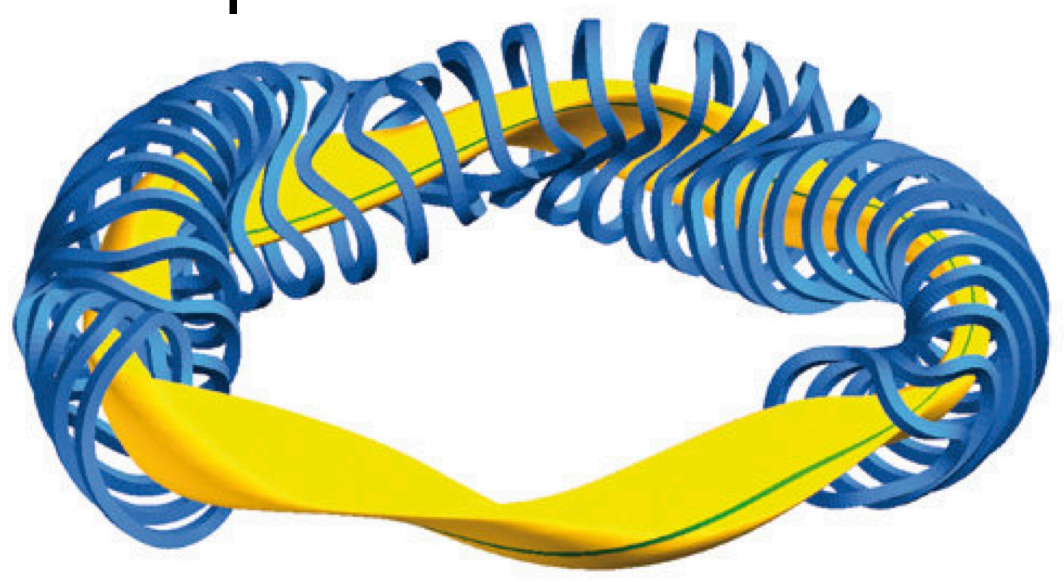
Fluid models (moments, conservation equations)

Kinetic models (distribution functions)

Single-particle models (particle pushing)

Modelling hierarchies in plasma physics

Macroscopic



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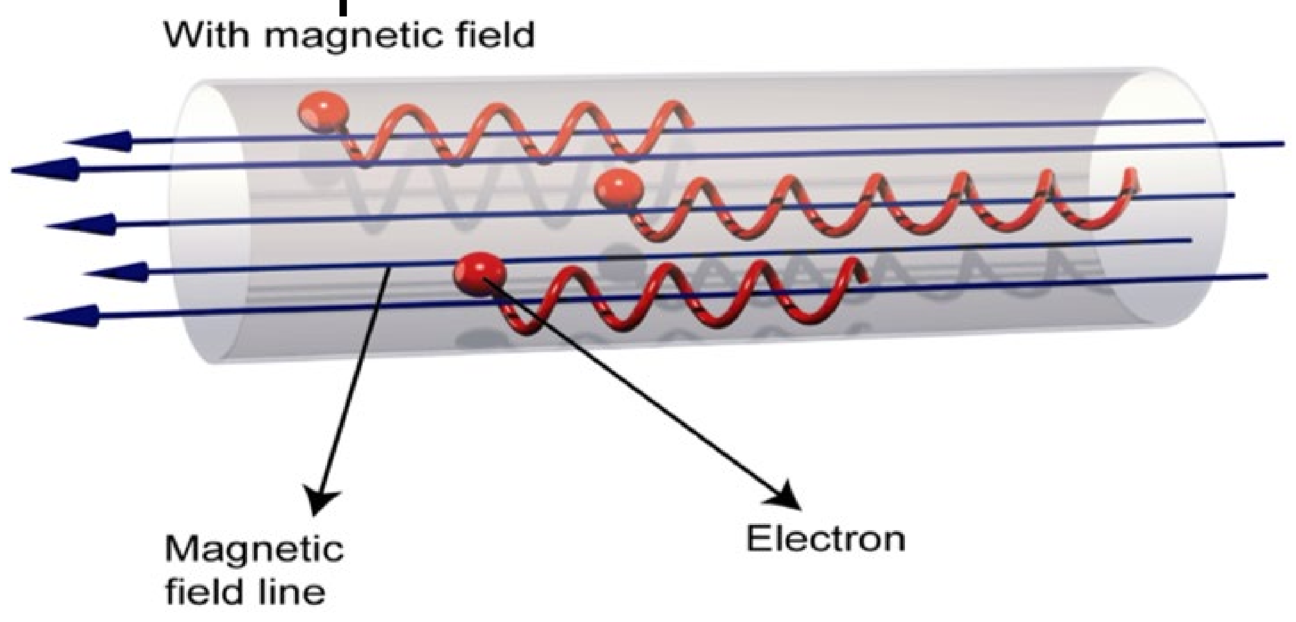


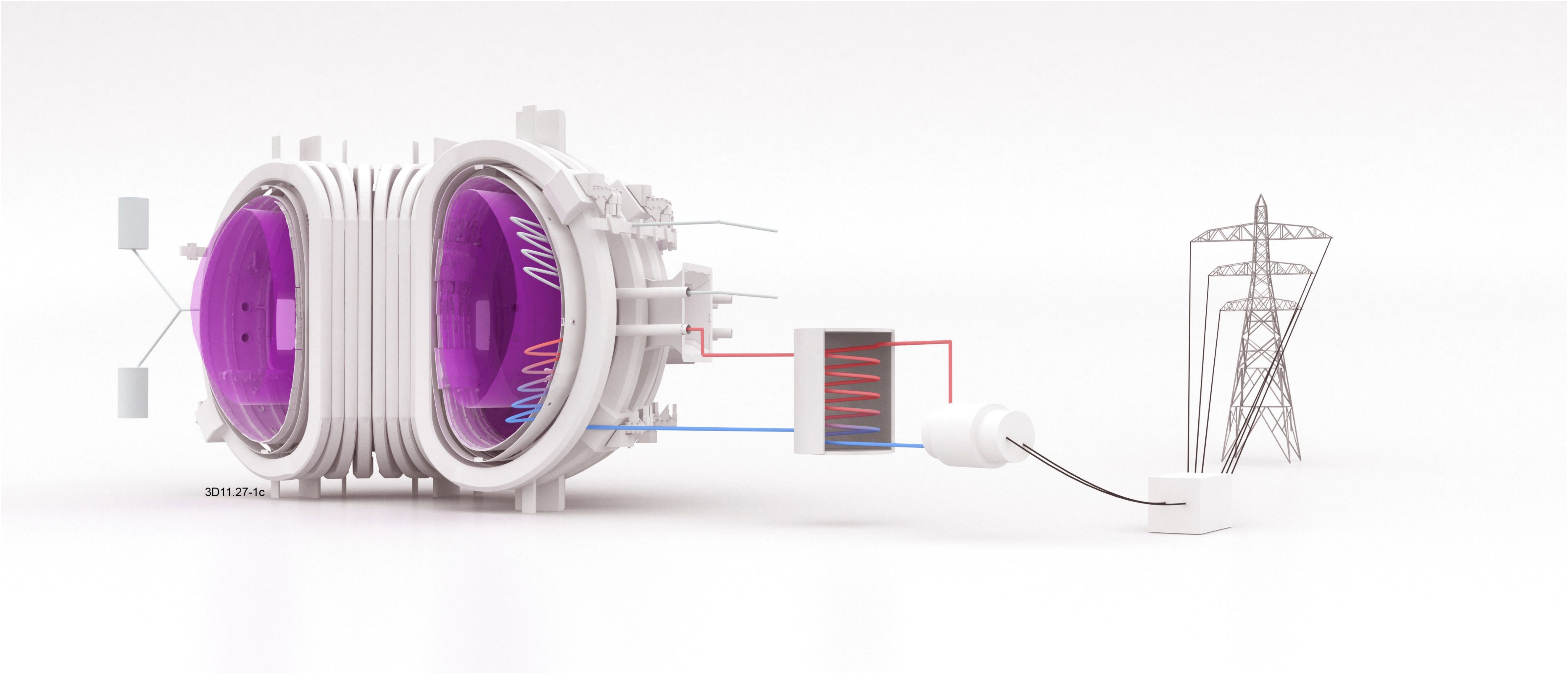
Fluid models (moments, conservation equations)

Kinetic models (distribution functions)

Single-particle models (particle pushing)

Magnetohydrodynamics (MHD)

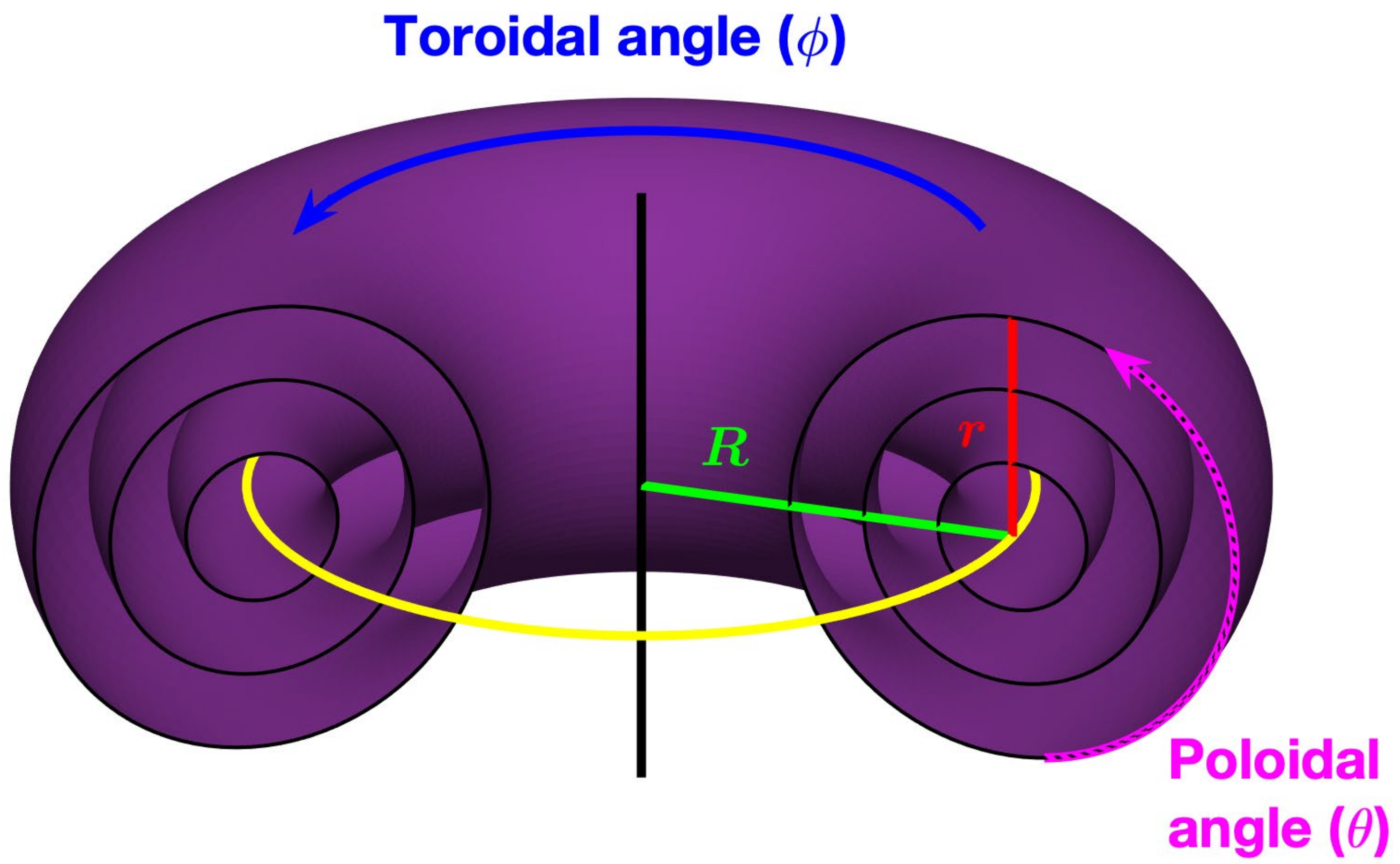
- A macroscopic description of plasmas in the continuum limit.
- A nonlinear dynamical system that is rich in spatio-temporal complexity.
- Important applications in the laboratory, space and astrophysics.



(Image credit: EUROfusion.)

MHD and (toroidal) magnetic confinement fusion

- MHD is used to describe the macroscopic behaviour of plasmas.
- Magnetic confinement fusion relies on steady-state operation and confinement.
- An important application of MHD is to understand and avoid large-scale instabilities.



(Figure from L.-M. Imbert-Gerard et al., arXiv:1908.05360 (2020).)

Toroidal current needed to create poloidally confining magnetic field → Current-driven instabilities

Ignition temperature (at $T = 15\ keV$): $(p\tau_E)_{min} \approx 8 atm.s$ → Pressure-driven instabilities

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- Magnetic confinement fusion relies on steady-state operation and confinement.
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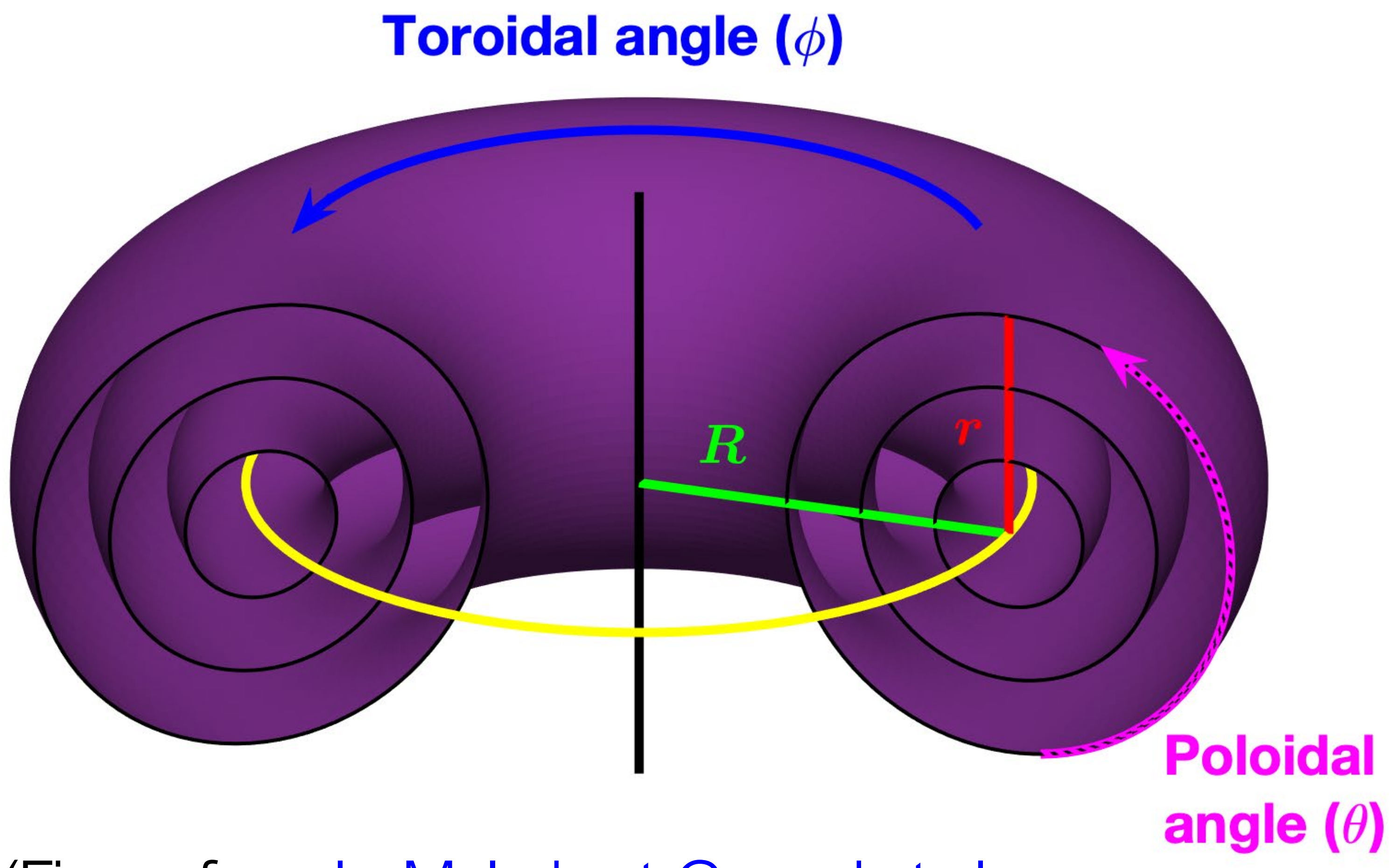
Can be studied with MHD



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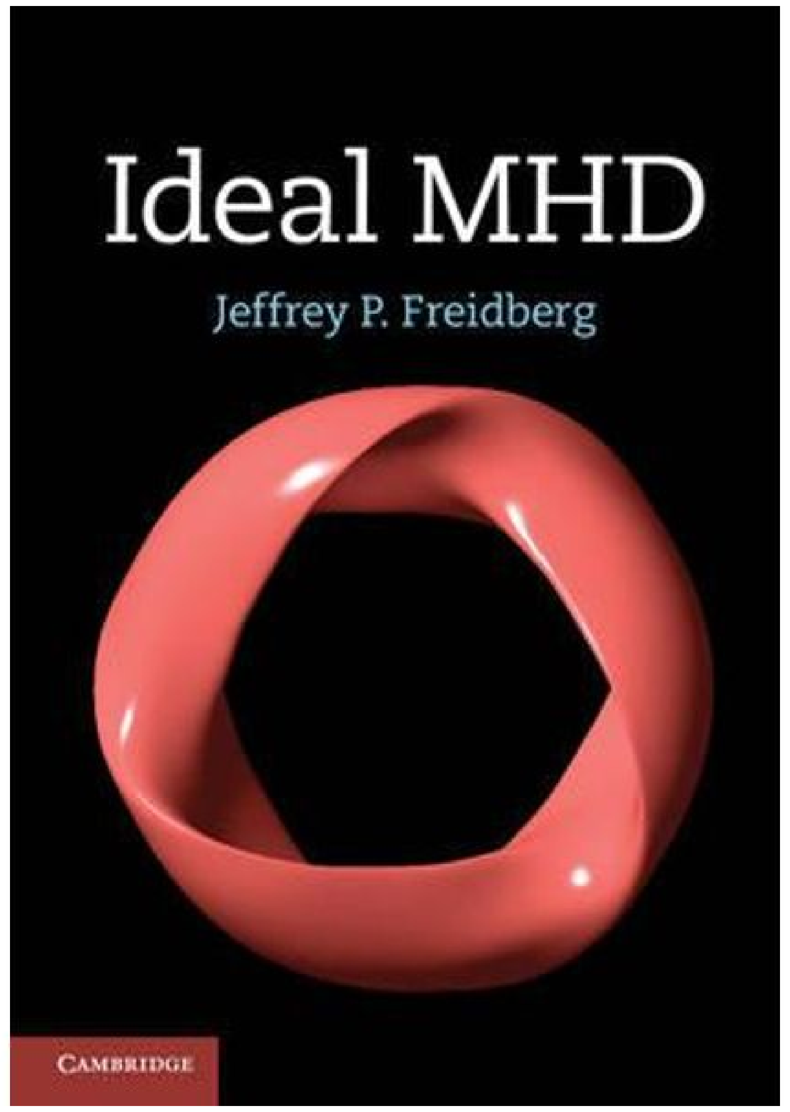
Recommended references



"Ideal MHD"

- J. P. Freidberg, Cambridge University Press (2014).
- Accessible and comprehensive introduction to ideal MHD for fusion.
- Covers the ideal MHD model, equilibrium and linear stability.

An Introduction to Stellarators
From magnetic fields to symmetries and optimization

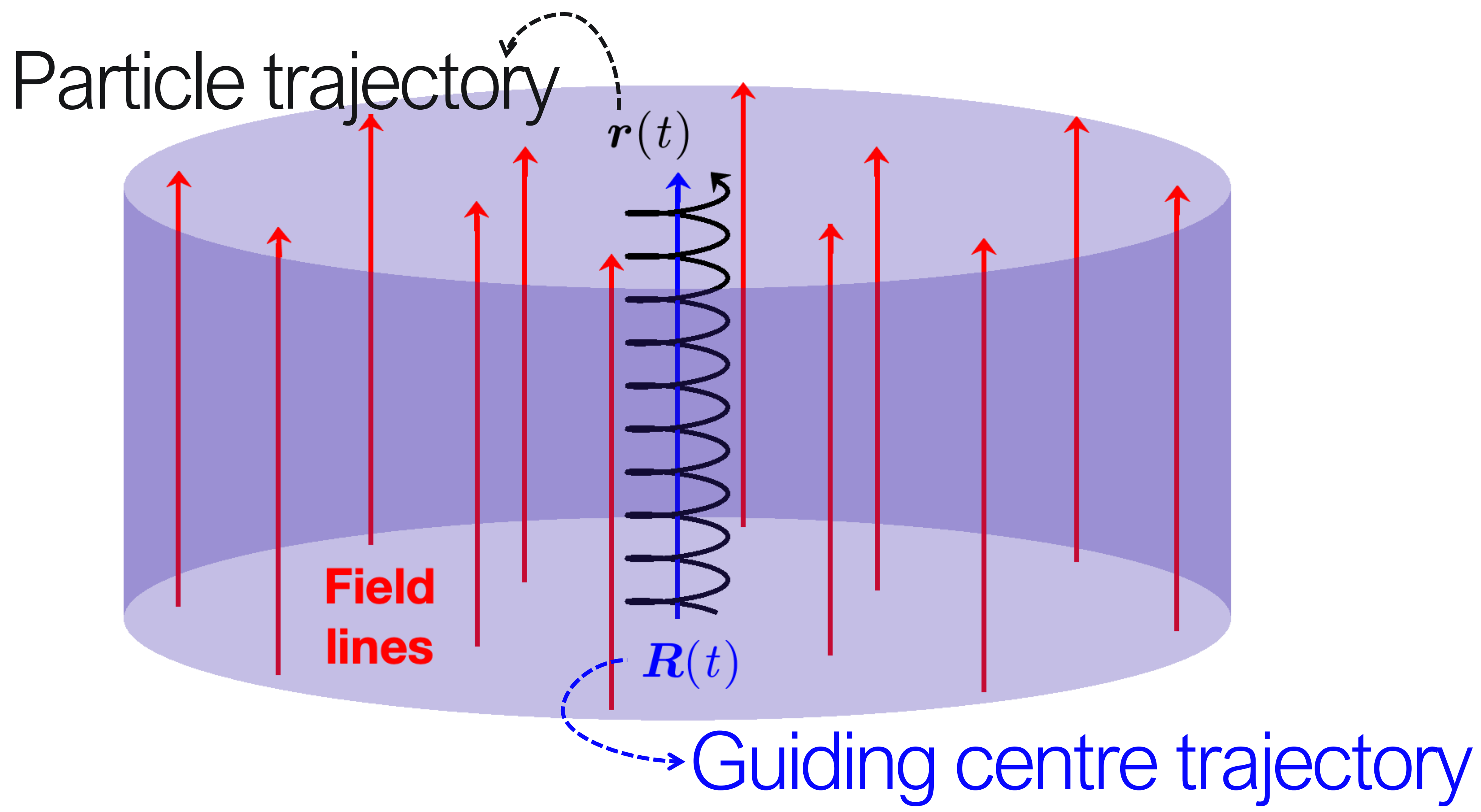
Lise-Marie Imbert-Gérard, Elizabeth J. Paul, Adelle M. Wright

"An Introduction to Stellarators: From magnetic fields to symmetries and optimization"

- L.-M. Imbert-Gerard, E. J. Paul and A. M. Wright (2020+).
- https://arxiv.org/abs/1908.05360
- A self-contained introduction covering the basic theoretical building blocks for modelling 3D magnetic fields, with applications to fusion device optimization and design.
- No physics background assumed.
- Coming soon(-ish) in book form.

An aside on the applicability of MHD

- Charged particles gyrate about magnetic field lines. If gyromotion ≪ scale lengths of the problem, this effect is unimportant.



(Figure from L.-M. Imbert-Gerard et al., arXiv:1908.05360 (2020).)

- In practice, this means MHD is generally applicable when:

Typical length scale	$a \sim 1m$	(Minor radius of device)
Typical time scale	$\tau_A \sim 2\mu s$	(For ideal MHD)
Typical velocity	$v_T \sim 500km/s$	(Ion thermal speed)

Model selection in plasma physics

- Nonlinearity and coupling across multiple spatial $(10^{-5} - 10^{3} m)$ and temporal $(10^{-12} - 10^{2} s)$ scales is a characteristic feature of plasma physics.
- Scale separation is the key underpinning principle when constructing and using plasma physics models.
- Assumptions abound! (E.g., the closure problem).
- In practice, the appropriate model depends on the specifics of the problem at hand.

Electromagnetics + hydrodynamics

=

Magnetohydrodynamics

Macroscopic description of plasmas: Electromagnetic fields

- In the MHD regime, we consider a fluid in an electromagnetic field.
- The electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields are governed by Maxwell's equations:

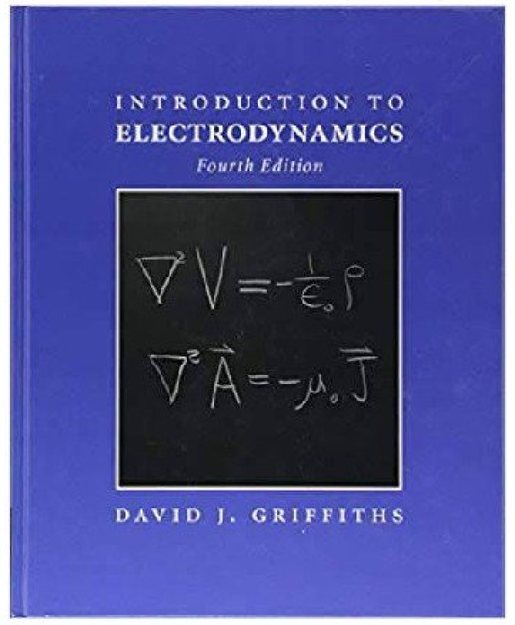
(Gauss' law) $$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

(Ampere's law) $$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \cancel{\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}}$$

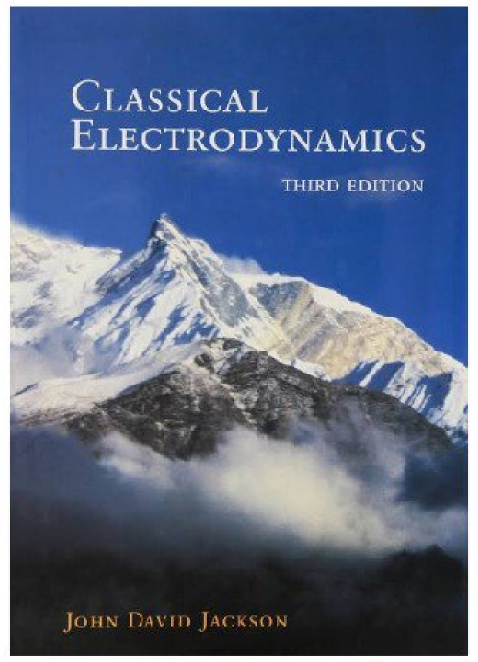
(Faraday's law) $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

Don't say you love the anime



If you haven't read the manga



- Here, $\mathbf{J}$ is the current density, ρ is the density, ϵ_0 is the vacuum permittivity, μ_0 is the vacuum permeability and c is the speed of light.

Macroscopic description of plasmas: Fluid conservation

- By taking successive moments of the Boltzmann equation (kinetic model), we can derive conservation equations for fluids.

- Since each plasma species (e.g., electrons and ions) has its own distribution function, we have conservation equations for each species s.

Mass continuity:

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0$$

Momentum conservation:

$$m_s n_s \left(\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \right) = q_s n_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) - \nabla \cdot \mathbf{P}_s + \mathbf{R}_s$$

n_s = density
$n_s \mathbf{v}_s$ = mean flow
$\mathbf{P}_s$ = pressure tensor
m_s = mass
q_s = charge
$\mathbf{R}_s$ = collisional momentum transfer

- But this is a complex system of equations, how can we simplify things?

Macroscopic description of plasmas: Single-fluid reduction

- In the MHD regime, the plasma is assumed to be quasi-neutral:

$$n_i = n_e$$

- The electron behaviour can be modelled by assuming $m_e \to \mathbf{0}$. This reduces the density to:

$$\rho \equiv \sum_s m_s n_s = m_i n_i$$

- The (mass) continuity equation reduces to:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- Combining the momentum conservation equations for each species:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{J} \times \mathbf{B} - \nabla \cdot (\mathbf{P}_i + \mathbf{P}_e)$$

- Where $\mathbf{J} = \sum_s q_s n_s \mathbf{v}_s$ is the current density and $\mathbf{v} = \mathbf{v}_i$ is the fluid velocity.

Macroscopic description of plasmas: Ohm's law

- The momentum conservation equation for electrons with $m_e \to 0$ gives us Ohm's law:

$$q_e n_e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla \cdot \mathbf{P}_e + \mathbf{R}_e = 0$$

- Which is usually written as:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}_e + \mathbf{R}_e}{en}$$

- The RHS of Ohm's law contains additional physics that appear in various extensions of the MHD model.

Macroscopic description of plasmas: Energy conservation

- The final component of the MHD model is the pressure.
- From the Boltzmann equation (kinetic model), each species has an energy conservation equation:

$$\frac{3n_s}{2}\left(\frac{\partial T_s}{\partial t} + \mathbf{v}_s \cdot \nabla T_s\right)_s + \mathbf{P}_s : \nabla \mathbf{v}_s + \nabla \cdot \mathbf{q}_s = Q_s$$

- Where T_s is the temperature, $\mathbf{q}_s$ is the heat flux and Q_s is collisional heating.
- We can define a total temperature and total pressure:

$$p = p_i + p_e = 2nT$$

$$T = (T_i + T_e)/2$$

- Now we need to couple the ion and electron energy equations.

Macroscopic description of plasmas: Pressure

- To couple the ion and electron energy equations, we make a series of assumptions which impose quantitative conditions on the formal validity of the MHD model.

- See Ch. 2.3.5 of Freidberg's Ideal MHD for details.

- Assuming that the energy equilibration time is small compared to the time scale of interest:

$$T_i \approx T_e = T$$
$$p_i \approx p_e = p/2$$

- The energy equation reduces to:

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma}\nabla_\parallel \cdot [(\kappa_{\parallel i} + \kappa_{\parallel e})\nabla_\parallel T]$$

- Where $\gamma = 5/3$ and $\kappa_\parallel$ is the parallel thermal conductivity coefficient.

Combining electromagnetic fields and conservation equations

- The basic (non-ideal) single-fluid MHD model:

Maxwell's equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

Fluid conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{J} \times \mathbf{B} - \nabla \cdot (\mathbf{P}_i + \mathbf{P}_e)$$

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What can we do with it?

Combining electromagnetic fields and conservation equations

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Ohm's law:

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This is still a pretty complicated system of equations.

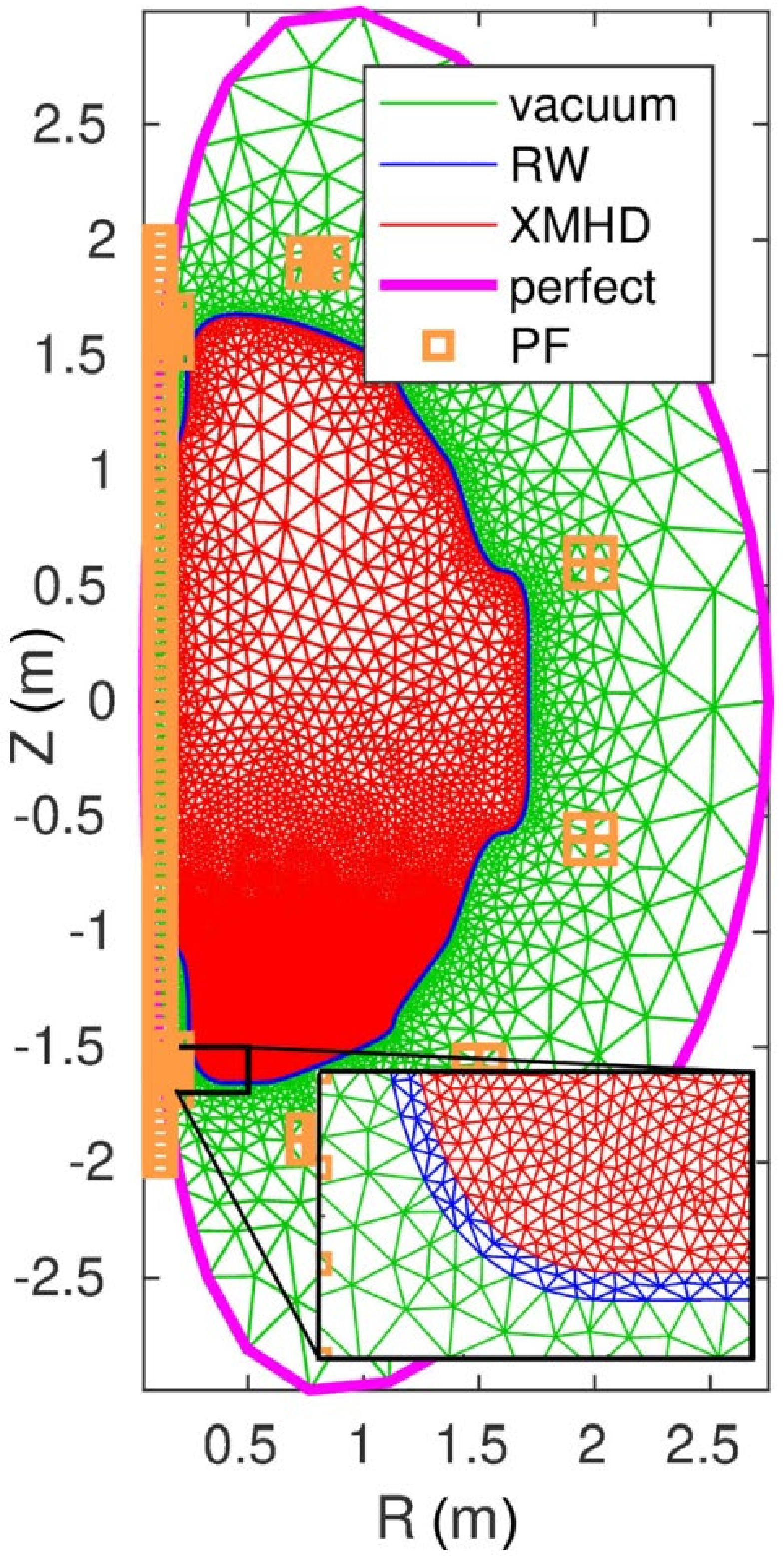
What can we do with it?

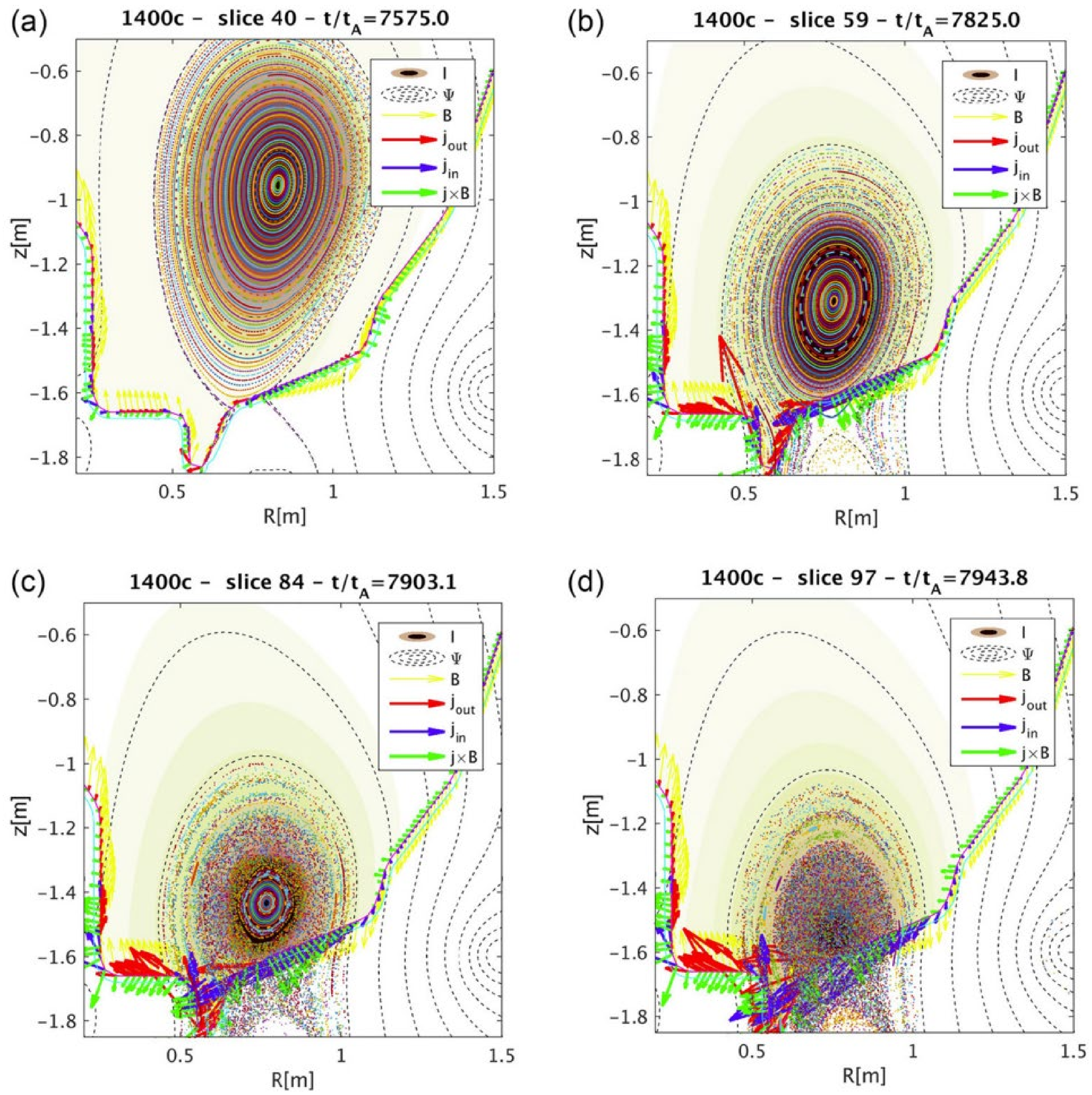
Give it to a computer.

Example: Extended-MHD modelling of fusion plasmas

- State-of-the-art codes are used to perform high-fidelity simulations of fusion plasmas.
- Examples include M3D-C1 (PPPL), NIMROD (U Wisc.-Madison) and JOREK (IPP, Garching).

Vertical displacement events happen when vertical control of a tokamak plasma is lost. The plasma rapidly moves upward or downward into the inner walls of the confinement vessel, leading to a disruption. VDEs can also cause large heat loads and electromagnetic stresses on the vessel.





Figures taken from D. Pfefferlé et al., Physics of Plasmas 25, 056106 (2018) [M3D-C1].

Figure 5. Poincaré plots at a sequence of times in a 3D M3D-C1 simulation an actual NSTX discharge that went vertically unstable. Red and blue arrows show halo currents into and out of the wall, respectively. Green arrows show the direction of the poloidal electromagnetic force density on the wall. Reprinted from [28], with the permission of AIP Publishing.

Reduction to the visco-resistive MHD model

Further reductions to the MHD model

- Further simplifications to the non-ideal single-fluid MHD model are possible:

Maxwell's equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

Fluid conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

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$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma} \nabla_\parallel \cdot [(\kappa_{\parallel i} + \kappa_{\parallel e}) \nabla_\parallel T]$$

Ohm's law:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}_e + \mathbf{R}_e}{en}$$

By making additional assumptions, these terms can be simplified.

Simplifying Ohm's law

- Recall the non-ideal Ohm's law:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}_e + \mathbf{R}_e}{en}$$

- The electron pressure tensor contains an isotropic (scalar) and anisotropic (tensor) term:

$$\mathbf{P}_e = p_e \mathbf{I} + \mathbf{\Pi}_e$$

- The effect of viscosity ($\mathbf{\Pi}_e$) is small compared to electron diamagnetic drift (∇p_e) which is comparable to the Hall effect ($\mathbf{J} \times \mathbf{B}$).

- The dominant contribution to $\mathbf{R}_e$ is electrical resistivity (η):

$$\frac{\mathbf{R}_e}{en} \sim \eta \mathbf{J}$$

- If the length scale of interest is large compared to the ion gyroradius, then $\nabla p_e / en$ is small compared to $\mathbf{v} \times \mathbf{B}$ which reduces Ohm's law to:

$$\boxed{\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}}$$

Simplifying momentum conservation (with viscosity)

- Recall the momentum equation:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{J}\times\mathbf{B} - \nabla\cdot\mathbf{P}$$

- Like the electron pressure tensor, we can decompose $\mathbf{P} = \mathbf{P}_i + \mathbf{P}_e$ into an isotropic and anisotropic component:

$$\mathbf{P} = (p_i + p_e)\mathbf{I} + \mathbf{\Pi}_i + \mathbf{\Pi}_e$$

- Since $p_e \approx p_i$ and $\mathbf{\Pi}_e$ is negligible by assumption:

$$\mathbf{P} = p\mathbf{I} + \mathbf{\Pi}_i$$

Contains viscosity terms

- When $\nabla\cdot\mathbf{v} \approx \mathbf{0}$, we can write:

$$\mathbf{P} = p\mathbf{I} - \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$$

$\propto$ Rate of deformation tensor

- A simplified model for momentum conservation with viscous effects (μ):

$$\boxed{\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{J}\times\mathbf{B} - \nabla p + \mu\nabla^2\mathbf{v}}$$

Equation of state: Simplifying the energy equation

- Under the preceding assumptions, the energy conservation equation reduces to:

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

- Which is referred to as the ideal MHD equation of state.

- Effects due to resistivity (η) can be modelled by modifying the equation of state:

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = (\gamma - 1)\eta J^2 + S_p$$

- To allow static ($\mathbf{v} = 0$) equilibria ($\partial_t \to 0$), a sink, S_p, is introduced.

Visco-resistive MHD

- The single-fluid MHD model with resistivity and viscosity:

Maxwell's equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

Fluid conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{J} \times \mathbf{B} - \nabla p + \mu \nabla^2 \mathbf{v}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = (\gamma - 1)\eta J^2 + S_p$$

Ohm's law:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$$

The MHD induction equation

Deriving the induction equation

- Combining Ohm's law:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$$

- With Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- Eliminates $\mathbf{E}$ to yield the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}$$

- Which is an evolution equation for the magnetic field.

The induction equation explained

- The induction equation can be expanded as:

$$\underbrace{\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{B}}_{\text{The convective derivative: } \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla} = \overbrace{(\mathbf{B} \cdot \nabla)\mathbf{v}}^{\text{Advection of } \mathbf{v} \text{ by } \mathbf{B}.} - \mathbf{B}(\nabla \cdot \mathbf{v}) + \frac{\eta}{\mu_0}\nabla^2 \mathbf{B}$$

Includes variation of $\mathbf{B}$ in time and advection of $\mathbf{B}$ by $\mathbf{v}$.

- The magnetic field ($\mathbf{B}$) and fluid ($\mathbf{v}$) are closely coupled.

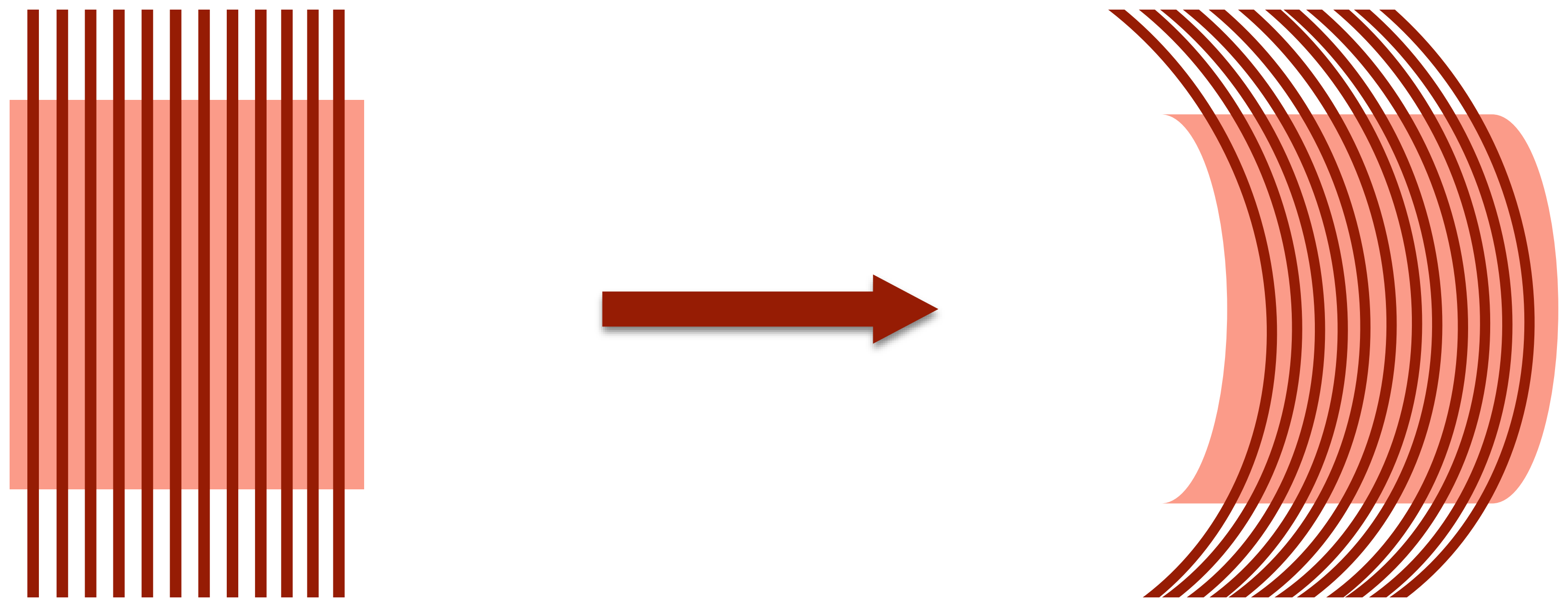
The ideal MHD limit

The ideal MHD limit

- When $\eta = 0$, the plasma is said to be ideal.
- The induction equation reduces to:

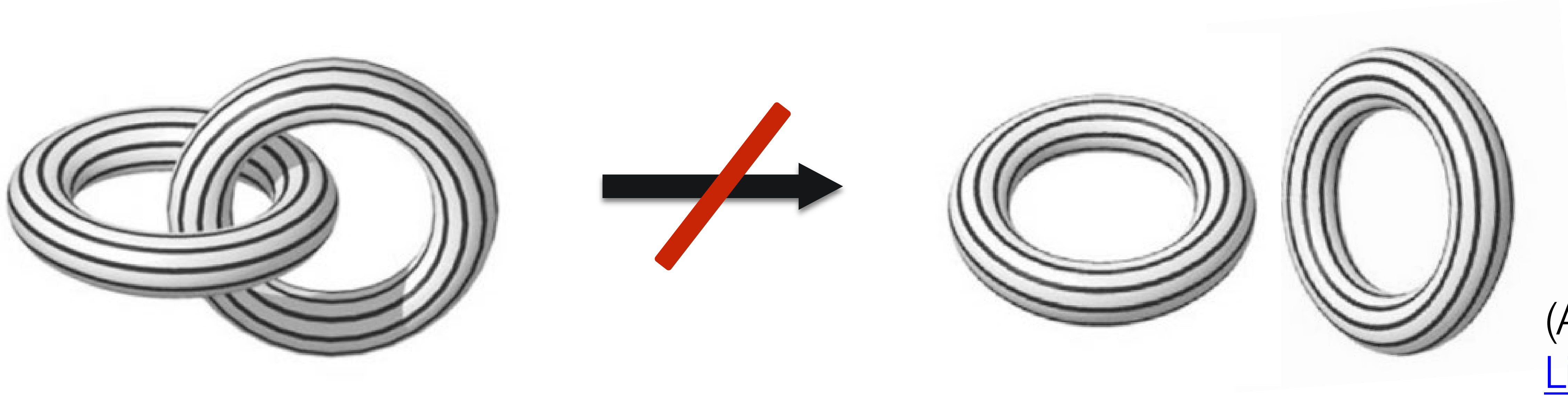
$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{v} - \mathbf{B}(\nabla\cdot\mathbf{v}) + \cancel{\frac{\eta}{\mu_0}\nabla^2\mathbf{B}}$$

- The magnetic field ($\mathbf{B}$) and fluid ($\mathbf{v}$) are exactly coupled.
- The magnetic field must move with the fluid. This is known as the frozen-in flux condition.



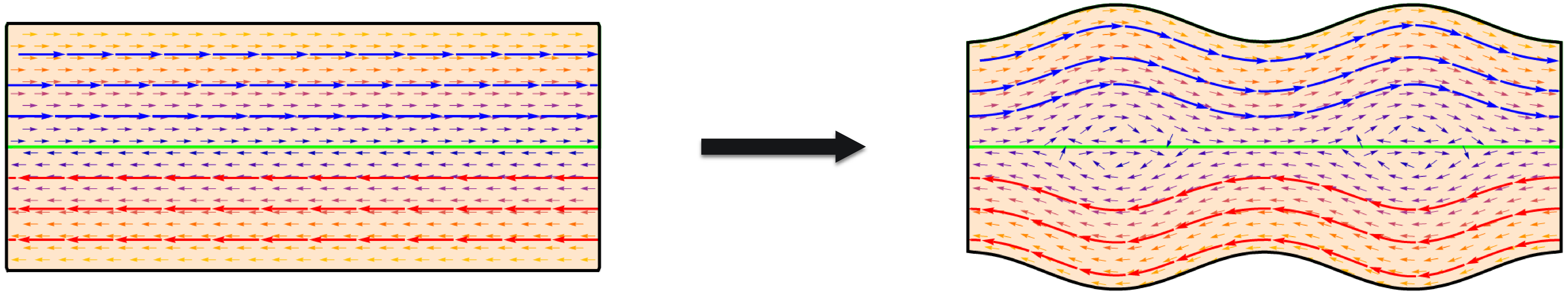
Implications of the frozen-in flux condition

- When the magnetic field is frozen-in, the connectivity of magnetic field lines cannot change.



(Adapted from Fig. 10 Wiegelmann & Sakurai, Living Reviews in Solar Physics 9.1 (2012))

- In ideal MHD, the topology of magnetic field lines is preserved exactly:



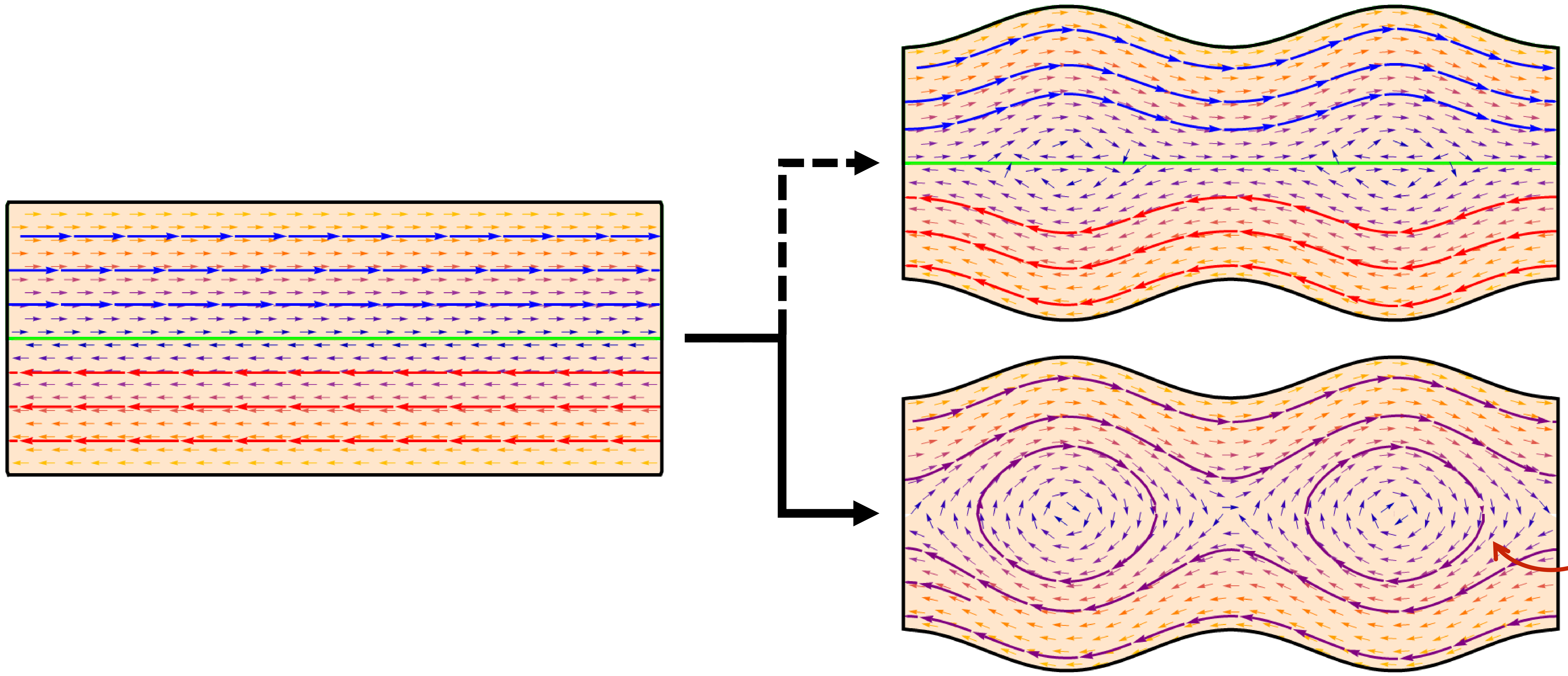
- Depending on the time scale of the problem of interest, $\eta = 0$ may **not** be a good approximation.

Magnetic reconnection

- When $\eta \neq 0$, the magnetic field can decouple from the fluid:

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) + \frac{\eta}{\mu_0}\nabla^2 \mathbf{B}$$

- This allows local changes in the connectivity of magnetic field lines via magnetic reconnection, leading to global changes in topology:



When $\eta \neq 0$, magnetic islands can form.

Example: Magnetic reconnection in the Earth's magnetosphere

The interaction between the solar wind and Earth's magnetic field is important for space weather, which can directly impact critical infrastructure on Earth.

Magnetic reconnection processes play an important role in this environment and is studied with numerical simulations as well as data from space missions, e.g., the Magnetospheric Multiscale Mission.

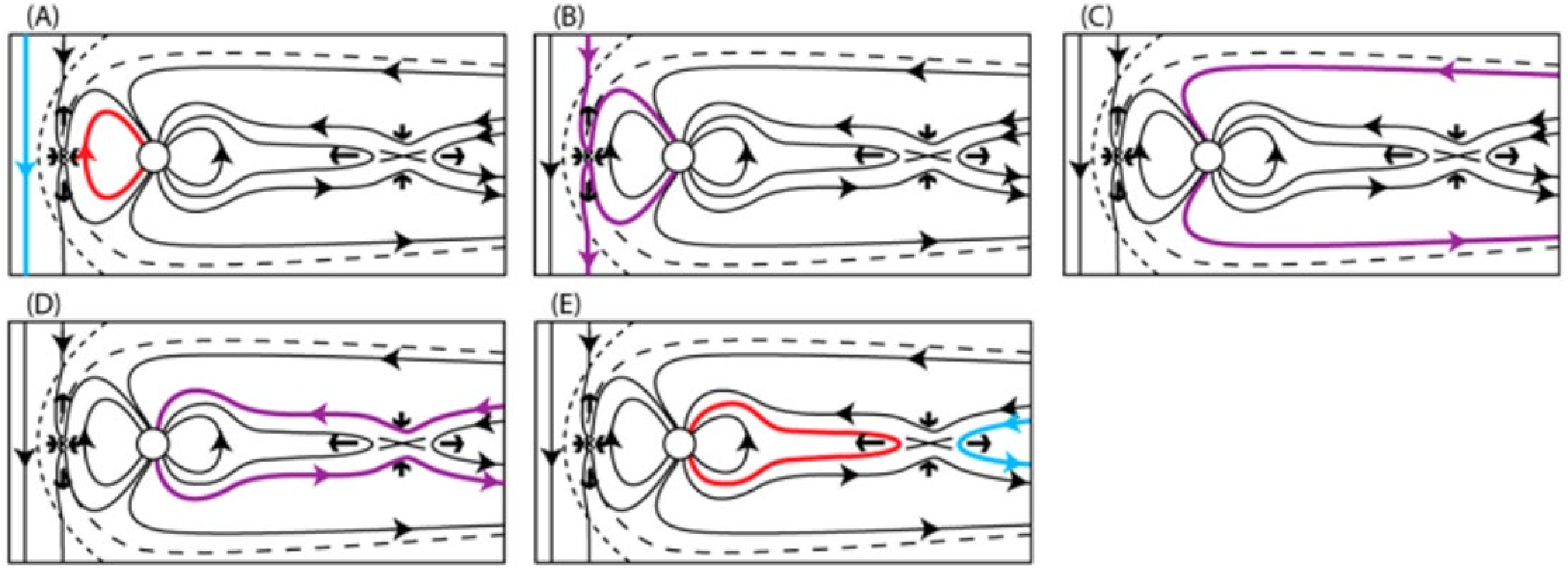


Fig. 8 Cartoon showing the progression of the Dungey cycle

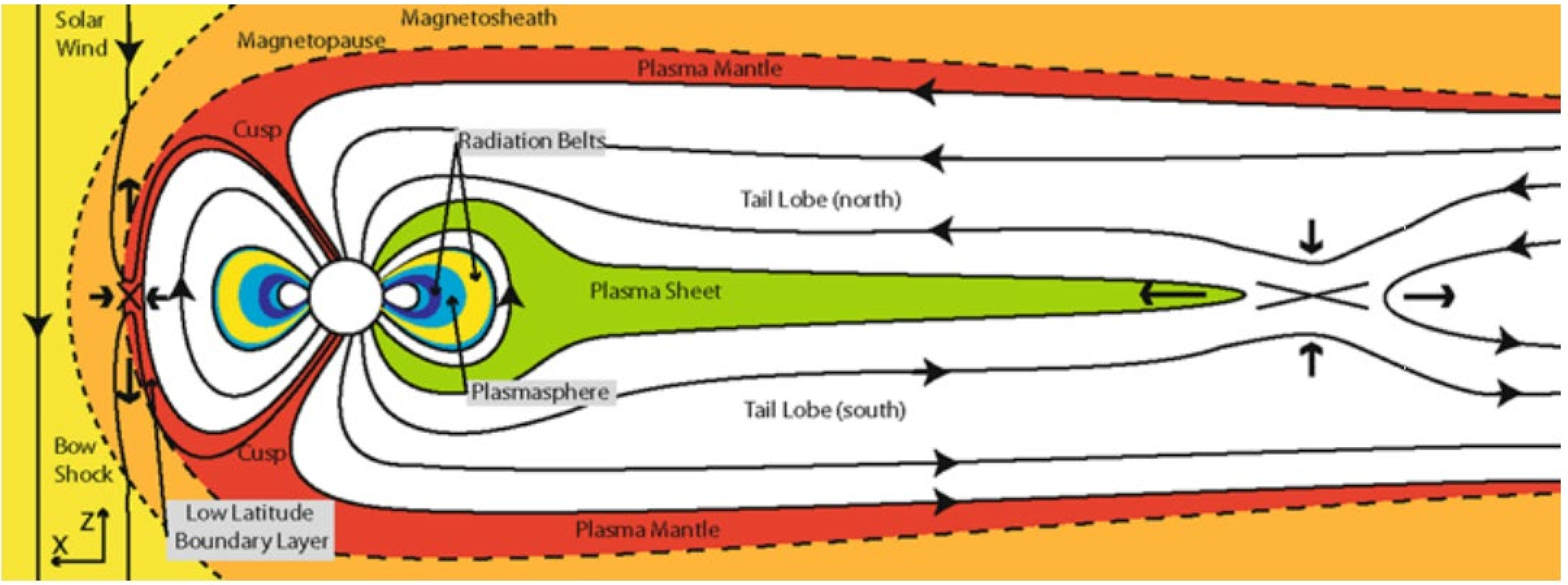
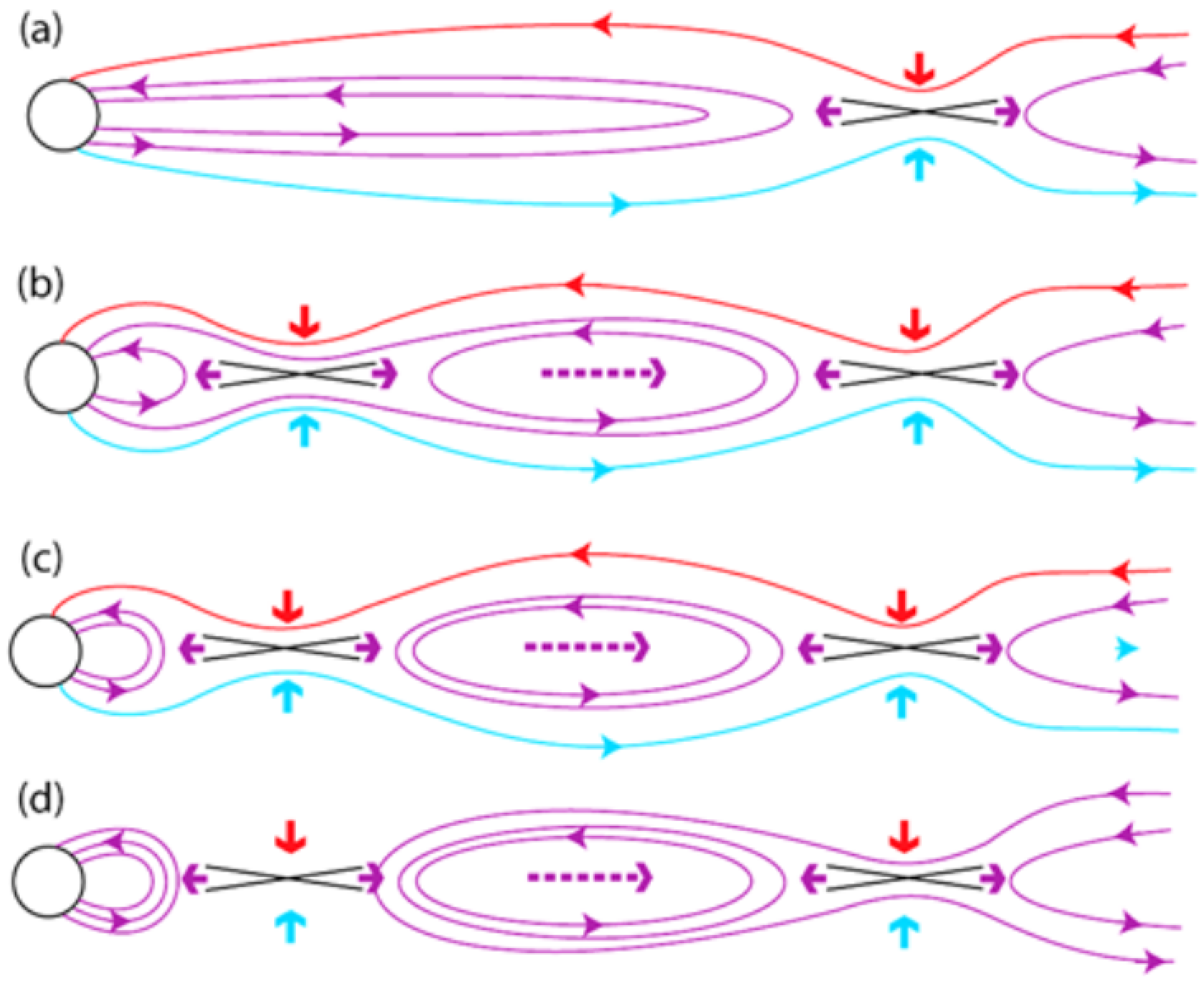


Fig. 1 A cartoon of the Earth's magnetosphere

Fig. 9 Cartoon showing the development of the near Earth neutral line and the formation of a plasmoid/flux rope which is released downtail once the near Earth neutral line enables reconnection between open field lines from the northern and southern lobe



Figures taken from J. P. Eastwood et al., Space Science Reviews 188.1 (2015).

Example: Sawtooth oscillations in tokamak plasmas

Under certain conditions, tokamak plasmas have been observed to exhibit periodic crashes in the electron temperature.

A leading model of the sawtooth phenomenon involves the formation of a large magnetic island, via reconnection, which displaces the magnetic axis.

Sawtooth crashes are observed on pretty much all tokamaks and continue to be the subject of robust debate and discussion.

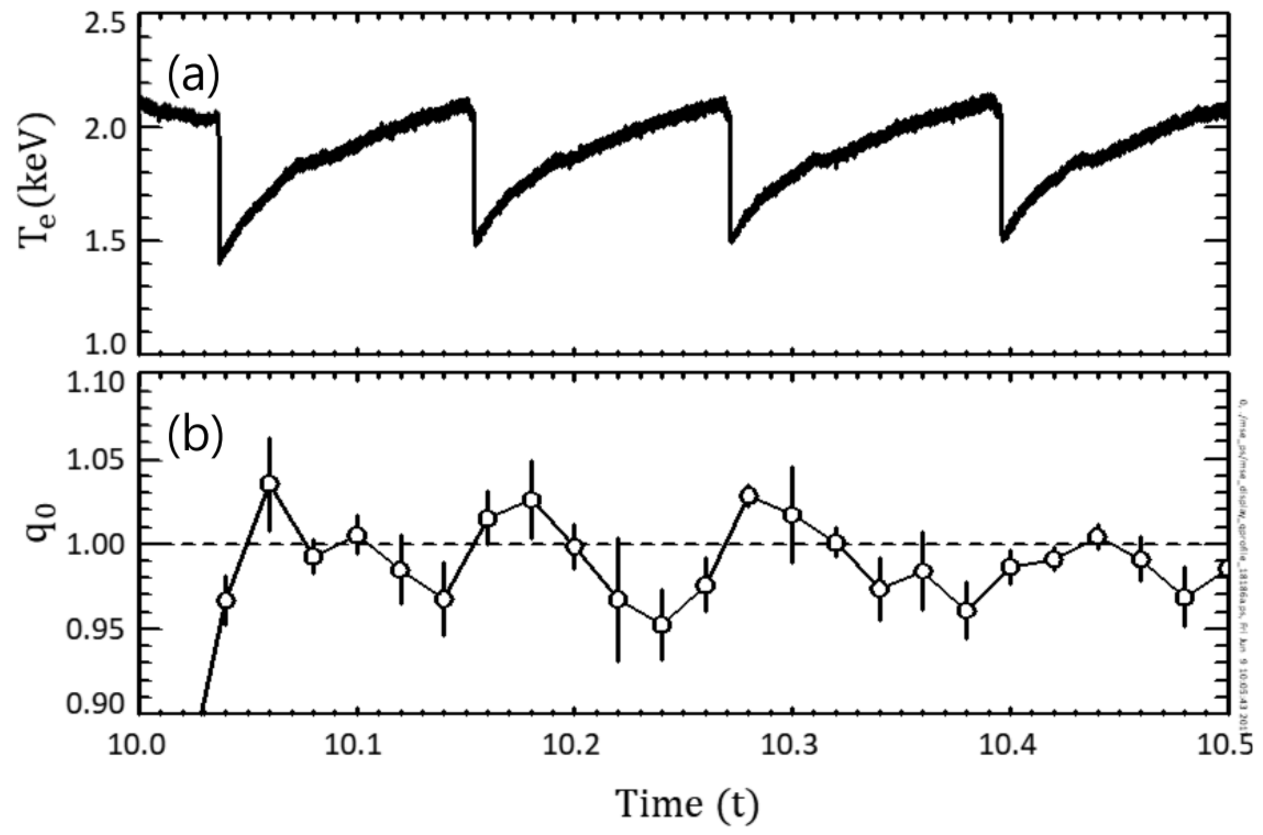
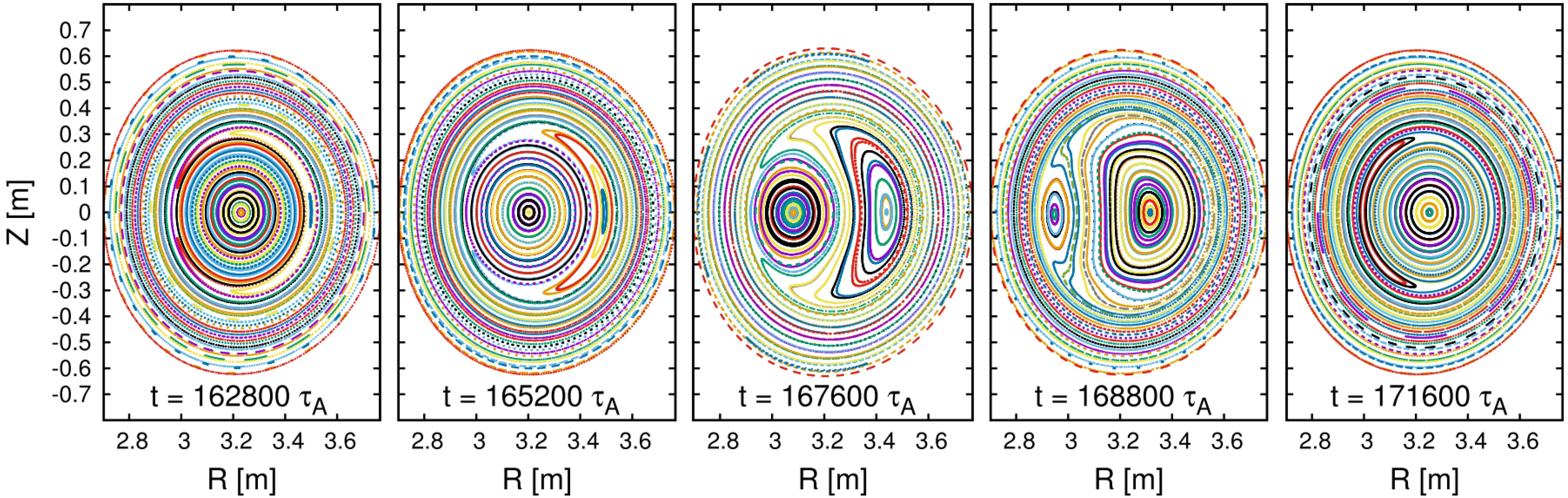


Figure 1. An example of the time evolution of (*a*) the central electron temperature and (*b*) corresponding q_0 in a sawtoothing discharge [shot #18186].



(Top) Figure taken from Y. B. Nam et al., Nuclear Fusion 58.6 (2018) [K-STAR].

(Left) Figure taken from I. Krebs et al., Physics of Plasmas 24.10 (2017) [M3D-C1].

FIG. 7. Poincaré plots showing the magnetic field line structure in the central plasma region at different points in time during a sawtooth cycle. As described in Kadomtsev's model, the $(m = 1, n = 1)$ magnetic island grows until it has entirely replaced the original plasma core. (Case "m0").

A remark on the ideal MHD "limit"

- When $\eta = 0$, the plasma is said to be ideal. Topological changes of the magnetic field are prohibited.
- When $\eta \neq 0$, the magnetic field can undergo reconnection locally, to change the global structure of $\mathbf{B}$.
- Conductivity is finite in any real system, so $\eta \neq 0$ in fusion plasmas.
- Proceed with caution when interpreting the limit $\eta \to 0$.
- However, the effect of η is local so well-established local techniques (e.g., boundary layer theory) can be used to handle small η.

Static ideal MHD equilibria

Reminder: the Ideal MHD model

- The single-fluid ideal MHD model:

Maxwell's equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

Fluid conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{J} \times \mathbf{B} - \nabla p$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0$$

Induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Reminder: the Ideal MHD model

- The single-fluid ideal MHD model.
- Let's now consider the static $(\mathbf{v} = 0)$ equilibrium $(\partial_t \to 0)$ limit:

Maxwell's equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

Fluid conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{J} \times \mathbf{B} - \nabla p$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0$$

Induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

The ideal MHD equilibrium model

- The ideal MHD equilibrium model describes static ($\mathbf{v} = 0$) equilibria ($\partial_t \to 0$) when $\eta = 0$:

Maxwell's equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

MHD force balance:

$$\mathbf{J} \times \mathbf{B} - \nabla p = 0$$

- Steady-state operation is important for magnetic confinement fusion.

- In the MHD regime, we are often interested in states where the plasma is not changing significantly on the time scale of interest.

- The ideal MHD equilibrium model can be a good approximation under these conditions.

Aside: The force-free MHD equilibrium model

- When the Lorentz force ($\mathbf{J} \times \mathbf{B}$) is negligible, the ideal MHD equilibrium model can be further simplified:

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$

- Which gives the nonlinear force-free equilibrium model:

$$\nabla \times \mathbf{B} = \alpha(\mathbf{x})\mathbf{B}$$
$$\mathbf{B} \cdot \nabla\alpha(\mathbf{x}) = 0$$

- And the linear force-free equilibrium model if α is constant.

- Force-free models are used in e.g., solar physics and some fusion applications.

Example: Force-free solar magnetic fields

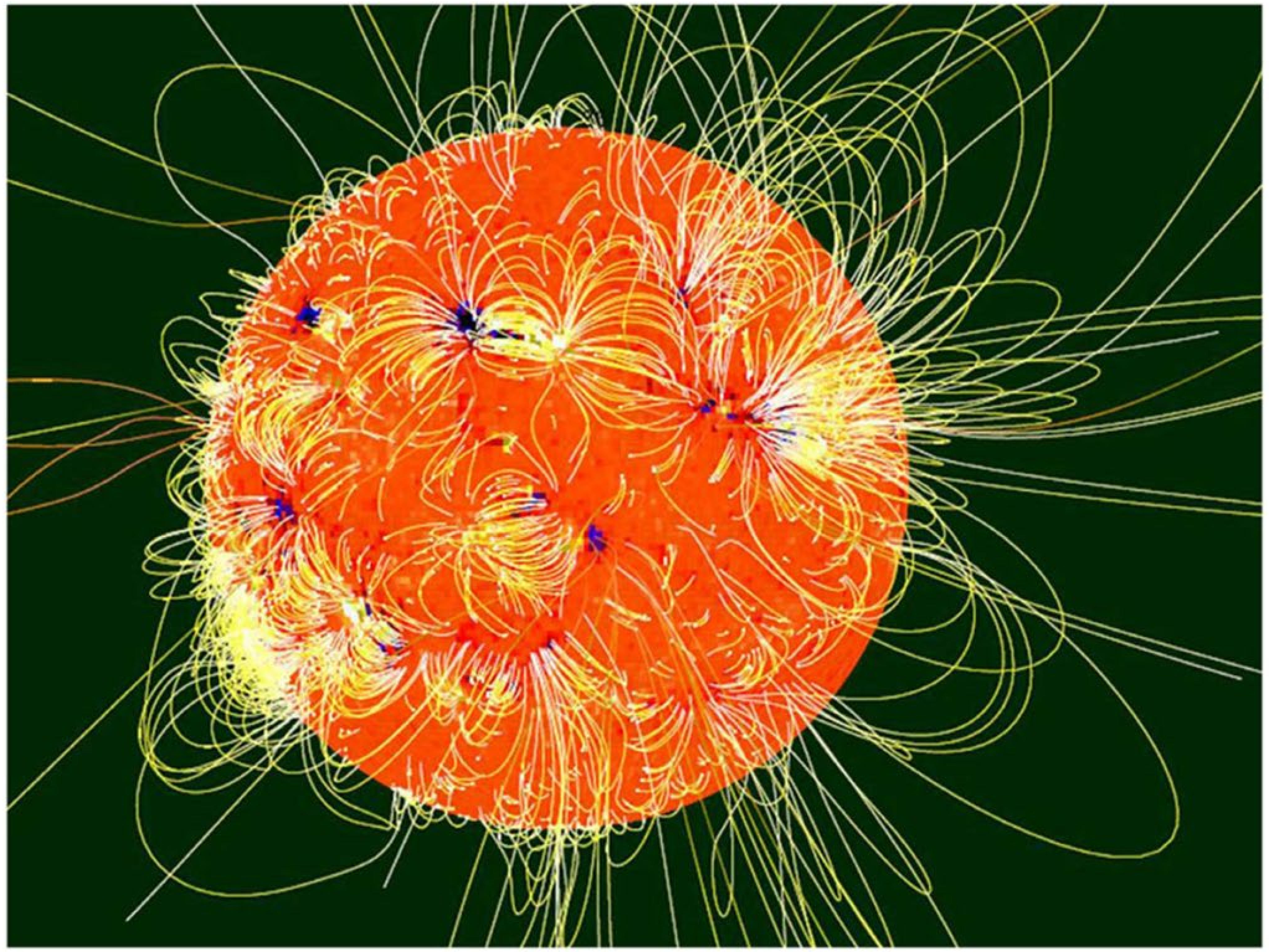


Fig. 3 Global potential field reconstruction. Image reproduced with permission from Wiegelmann and Solanki (2004), copyright by ESA

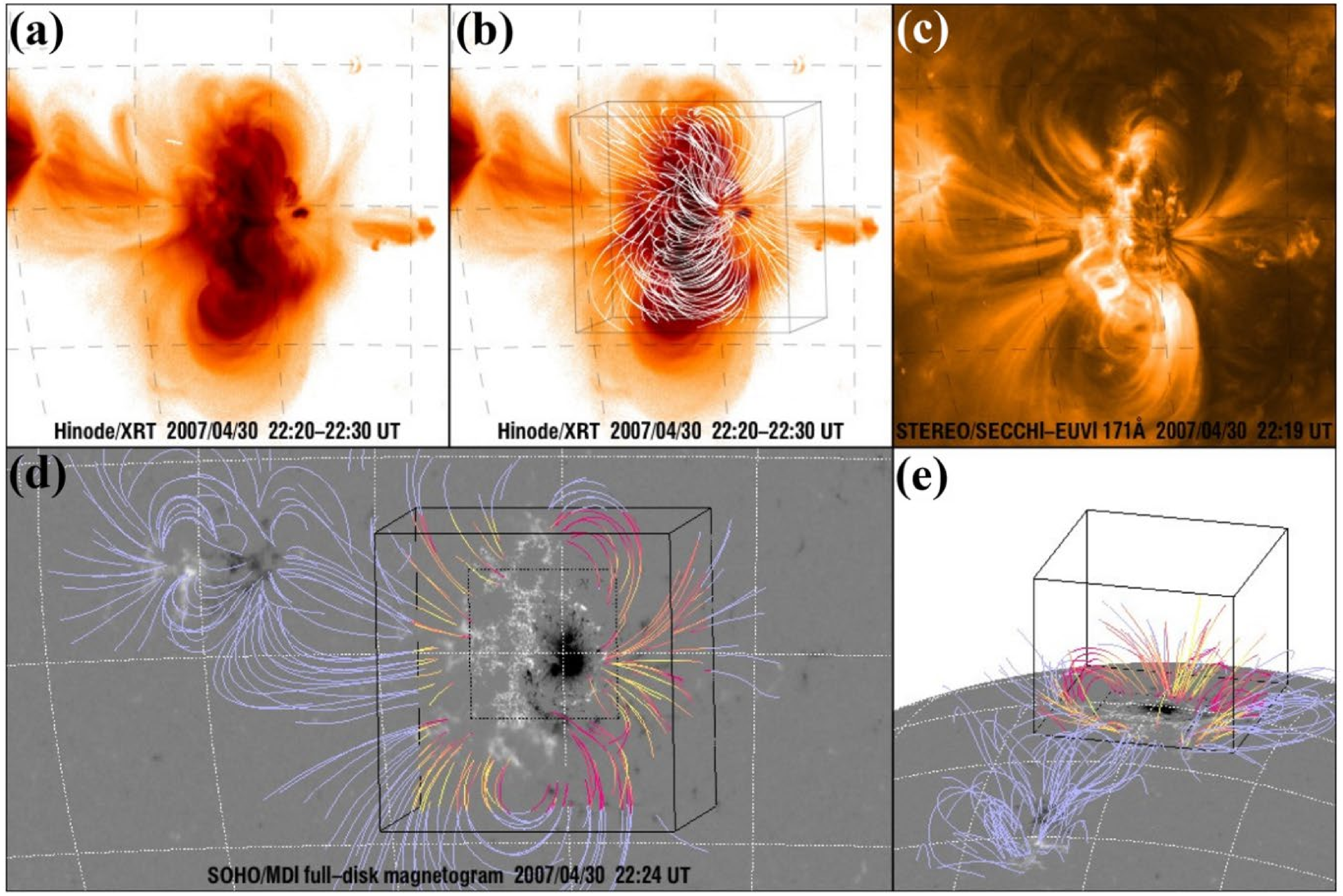


Fig. 15 **a–c** A series of coaligned images of active region AR 10953. In **b** field lines (white) from best fit NLFFF model are overlaid. **d**, **e** show the trajectories of loops from different viewpoints. The stereoscopically-reconstructed loops are taken from (Aschwanden et al. 2008b). The solid cube outlines the computational box of the NLFFF models. The interior dotted line outlines the FOV of Hinode. The STEREO-loops are coloured in blue outside the NLFFF-domain and are coloured with the misalignment angle ϕ of STEREO-loops and best fitting NLFFF model from yellow through orange to red with $5° \leq \phi \leq 45°$. Image reproduced with permission from Fig. 1 of DeRosa et al. (2009), copyright by AAS

Figures from [T. Wiegelmann & T. Sakurai, Living Reviews in Solar Physics 9.1 (2012)].

Aside: The ideal MHD equilibrium model via energy minimisation

- A common construct for deriving the ideal MHD force balance is to minimise potential energy:

$$W_{potential} = \int_{\Omega} \left(\frac{p}{\gamma - 1} + \frac{|\mathbf{B}|^2}{2\mu_0} \right) dv$$

- Using calculus of variations, $W_{potential}$ is stationary when:

$$\mathbf{J} \times \mathbf{B} - \nabla p = 0$$

MHD force balance equation

- Note that equilibrium does not necessarily imply energy minimum.
- Energy minimisation is the theoretical basis for several 3D MHD equilibrium codes (e.g., VMEC, SPEC).
- Energy 'minimisation' also depends critically on the choice of variations (i.e., what you are minimising with respect to). It gives the same equation, but the **physical** interpretation of the solution is nuanced.

Static ideal MHD equilibrium: 1D

- In cylindrical coordinates (r, θ, z) 1D MHD equilibria satisfy:

$$\frac{d}{dr}\left(p(r) + \frac{B_\theta^2(r) + B_z^2(r)}{2\mu_0}\right) + \frac{B_\theta^2(r)}{\mu_0 r} = 0$$

Plasma pressure | Magnetic pressure | Magnetic tension

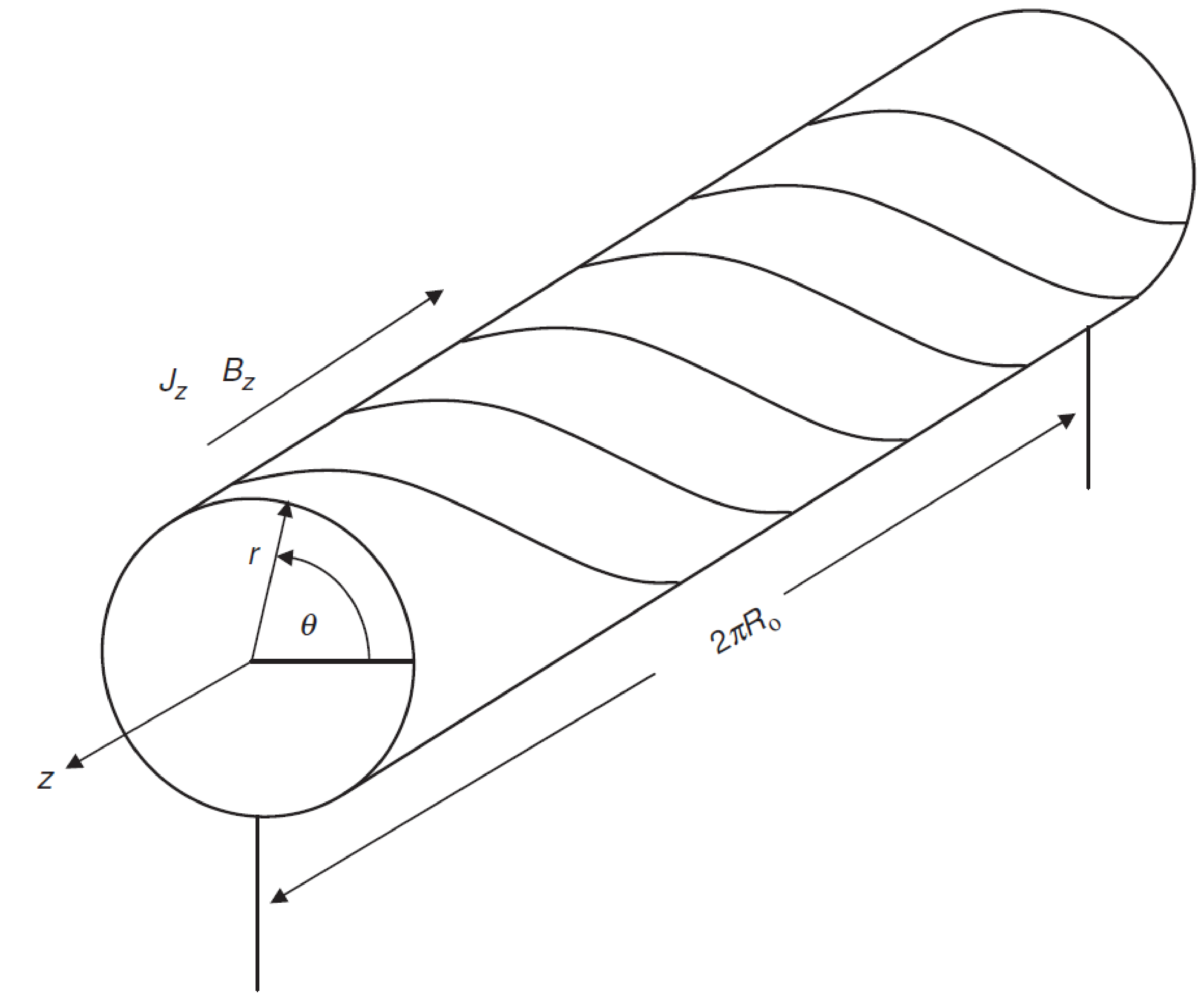


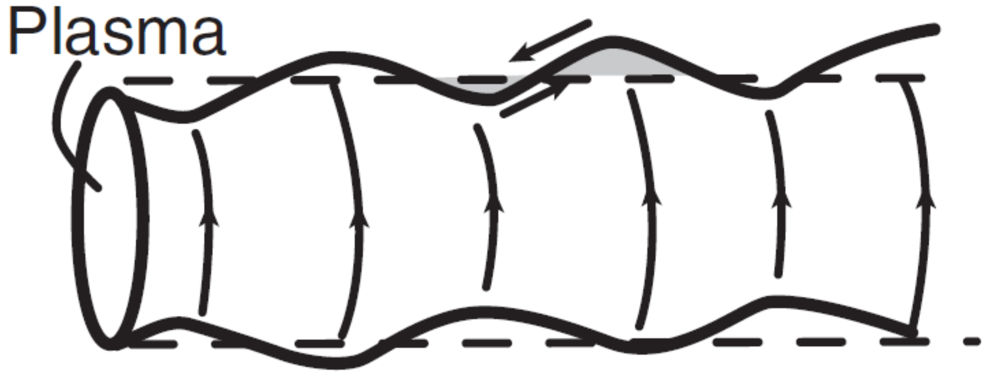
Figure 5.7 General screw pinch geometry.

Figures from J. P. Freidberg, Ideal MHD, CUP (2014).

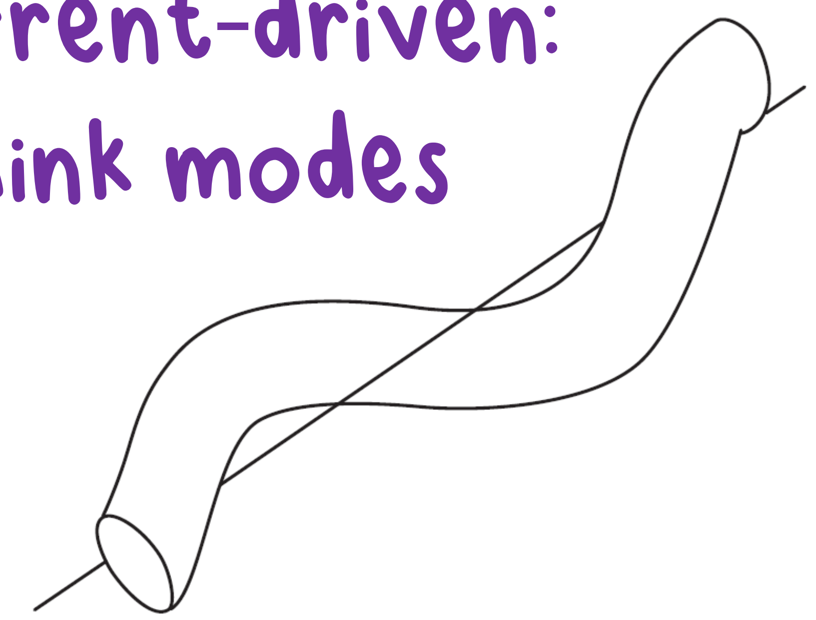
- Cylindrical geometry, sometimes referred to as a screw pinch.
- Analytically tractable so commonly used.
- Can model tokamaks in the large aspect ratio limit, but misses effects associated with toroidal curvature (e.g., poloidal mode coupling).

Some instabilities can be studied with the cylindrical model

Pressure-driven: Interchange modes



Current-driven: Kink modes



Static ideal MHD equilibrium: 2D

- 2D axisymmetric ($\partial_\phi \rightarrow 0$) equilibria are described by solutions of the Grad-Shafranov equation:

$$\underbrace{R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi(R,Z)}{\partial R}\right)+\frac{\partial^2\psi(R,Z)}{\partial Z^2}}_{\text{This operator is sometimes written as: } = \Delta^*\,\psi(R,Z)} = -\mu_0 R^2\frac{dp(\psi)}{d\psi}-\frac{1}{2}\frac{dF(\psi)^2}{d\psi}$$

- The free functions, $p(\psi)$ and $F(\psi) = RB_\phi$, are flux functions as they depend only on ψ and follow from:

$$\mathbf{B}\cdot\nabla p = 0$$

$$\mathbf{J}\cdot\nabla p = 0$$

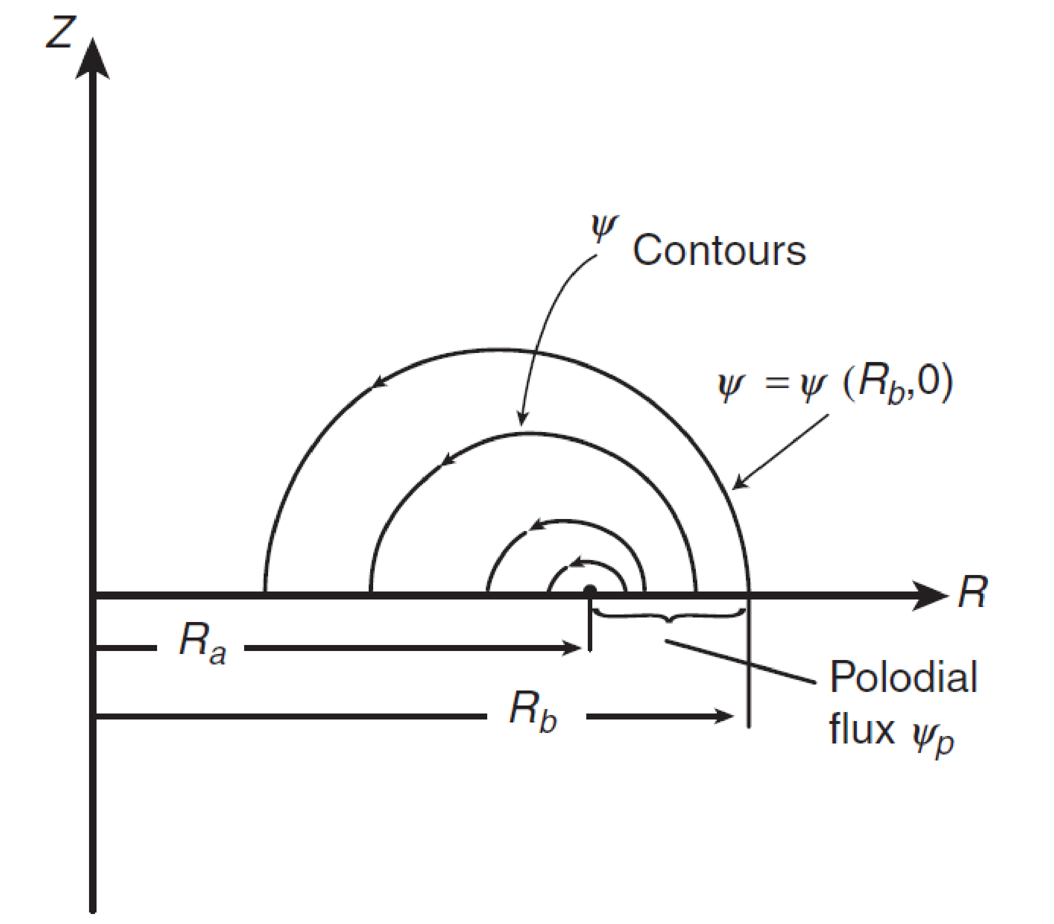


Figure 6.2 Washer-shaped surface through which the poloidal flux ψ_p passes.

Figure from: J. P. Freidberg, Ideal MHD, CUP (2014).

Example: Axisymmetric tokamak equilibrium

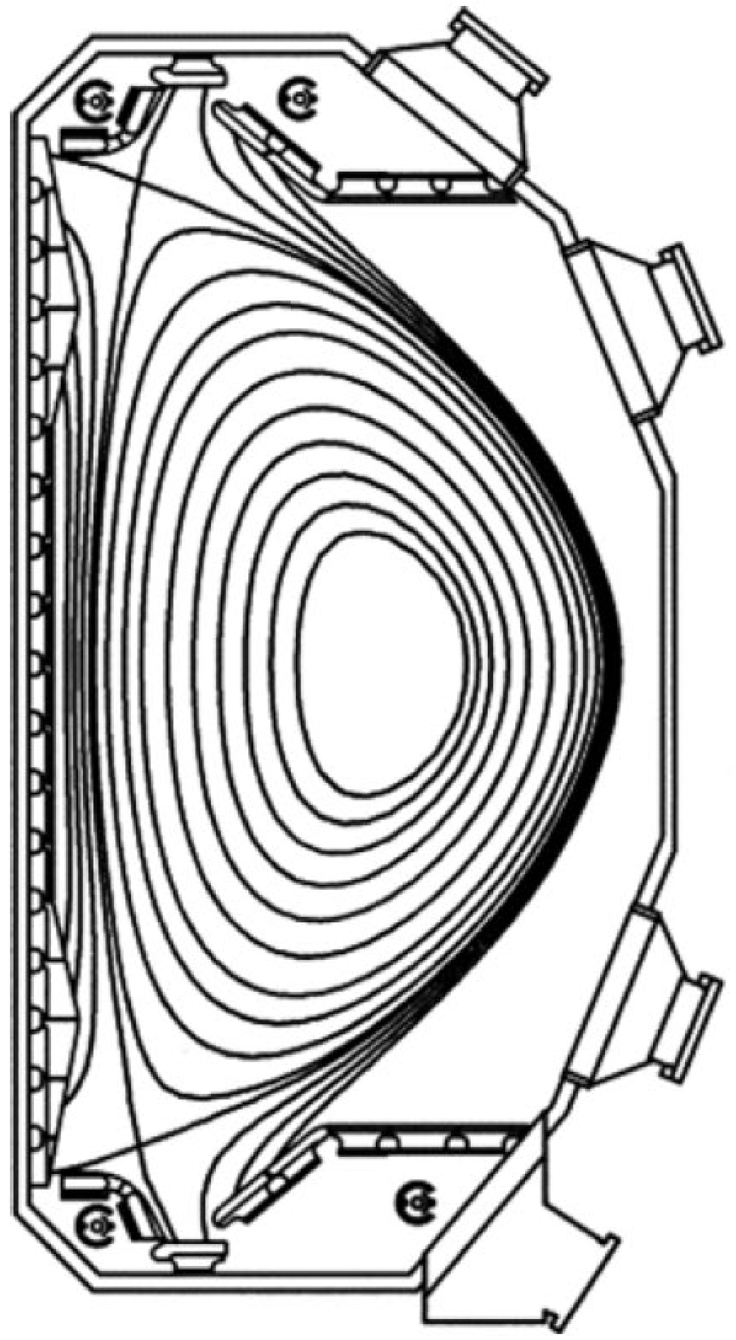


Figure 6.18 Numerically computed equilibrium for the DIII-D tokamak at General Atomics. Shown are flux surface plots corresponding to auxiliary heated high β tokamak operation. From DIII-D Team, 1998. Reproduced with permission from Elsevier.

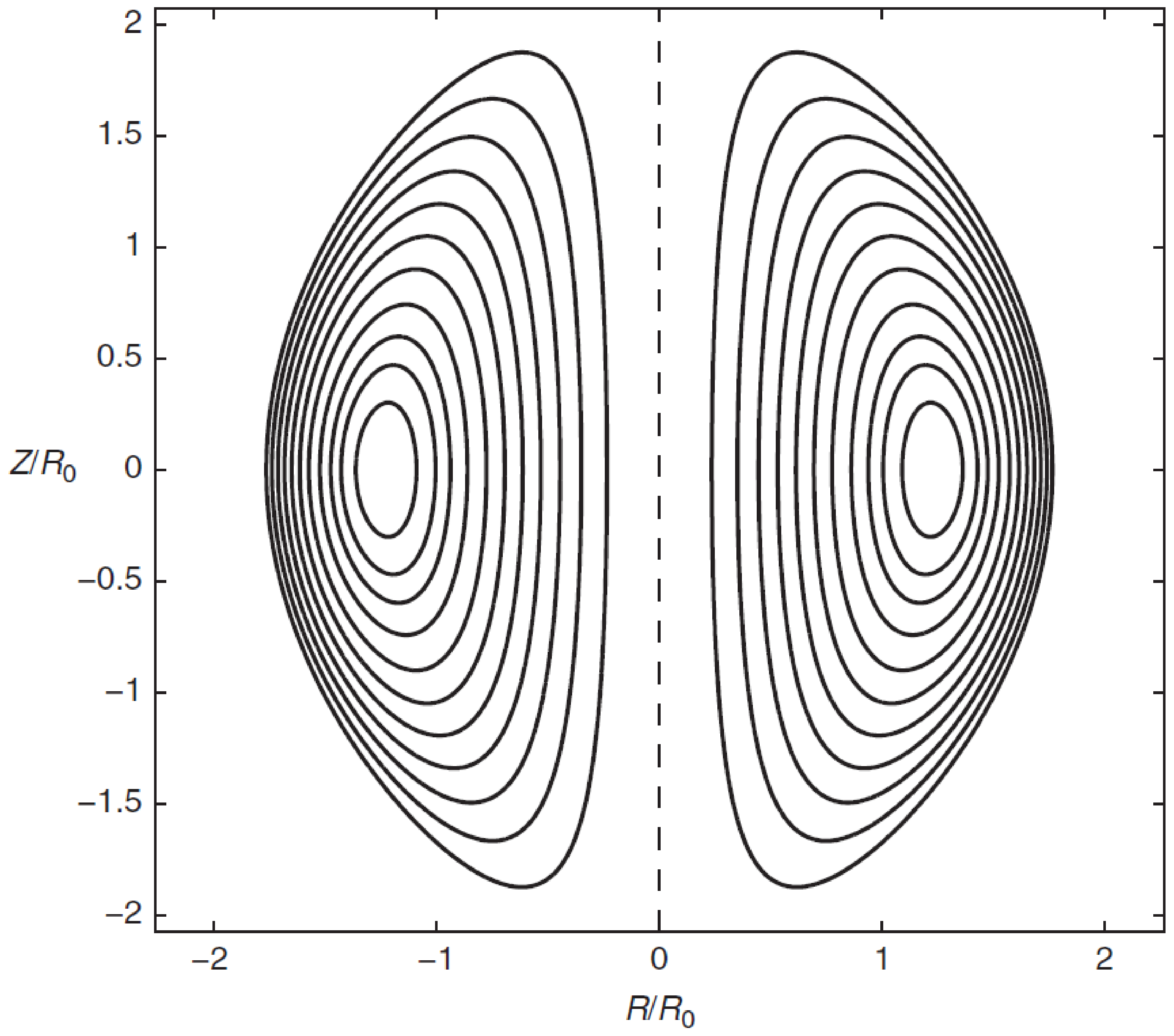


Figure 6.24 Exact Solov'ev equilibrium for MAST

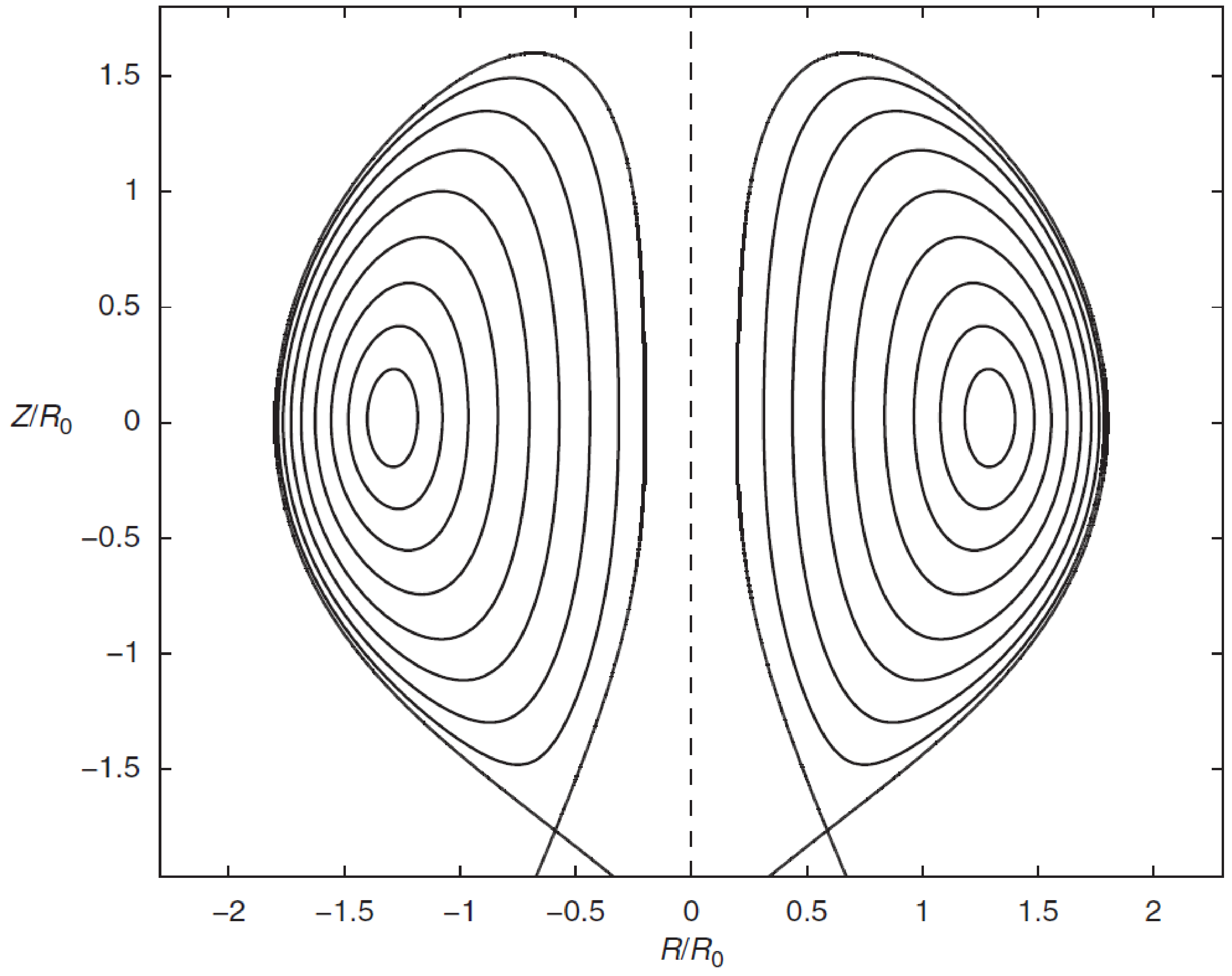


Figure 6.29 Exact single null divertor Solov'ev equilibrium for NSTX.

Figures from J. P. Freidberg, Ideal MHD, CUP (2014).

Static ideal MHD equilibrium: 3D

- Finding solutions for MHD equilibria when there is no continuous symmetry in the toroidal direction is an open topic of research.

- Independent of the MHD model, the structure of $\mathbf{B}$ is closely linked to Hamiltonian mechanics.

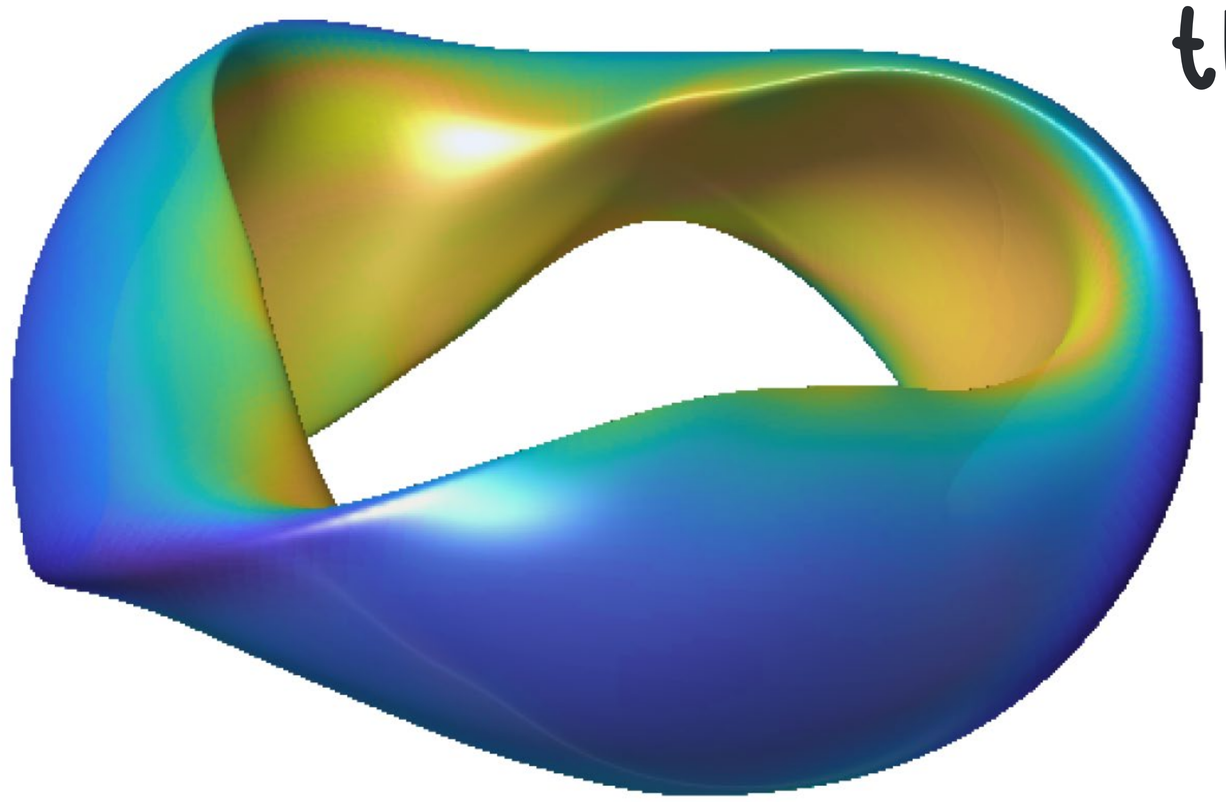
- When $\partial\phi \not\rightarrow \mathbf{0}$, new structures for $\mathbf{B}$ are possible. These are not guaranteed to be consistent with assumptions of the ideal MHD model.

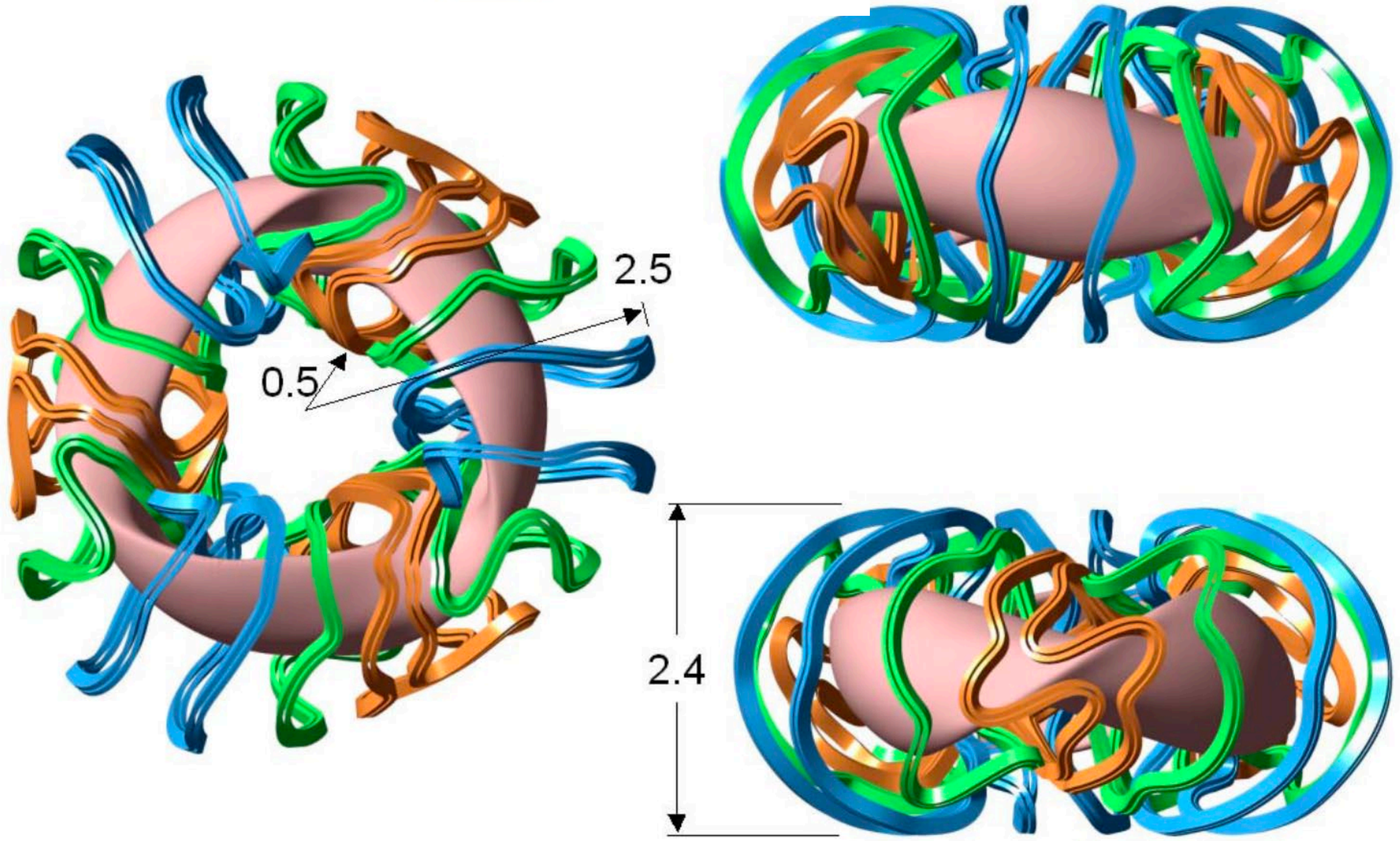
- Frontier challenges in 3D MHD are closely linked to dynamical systems theory.

- Stellarators are an example of non-axisymmetric devices.

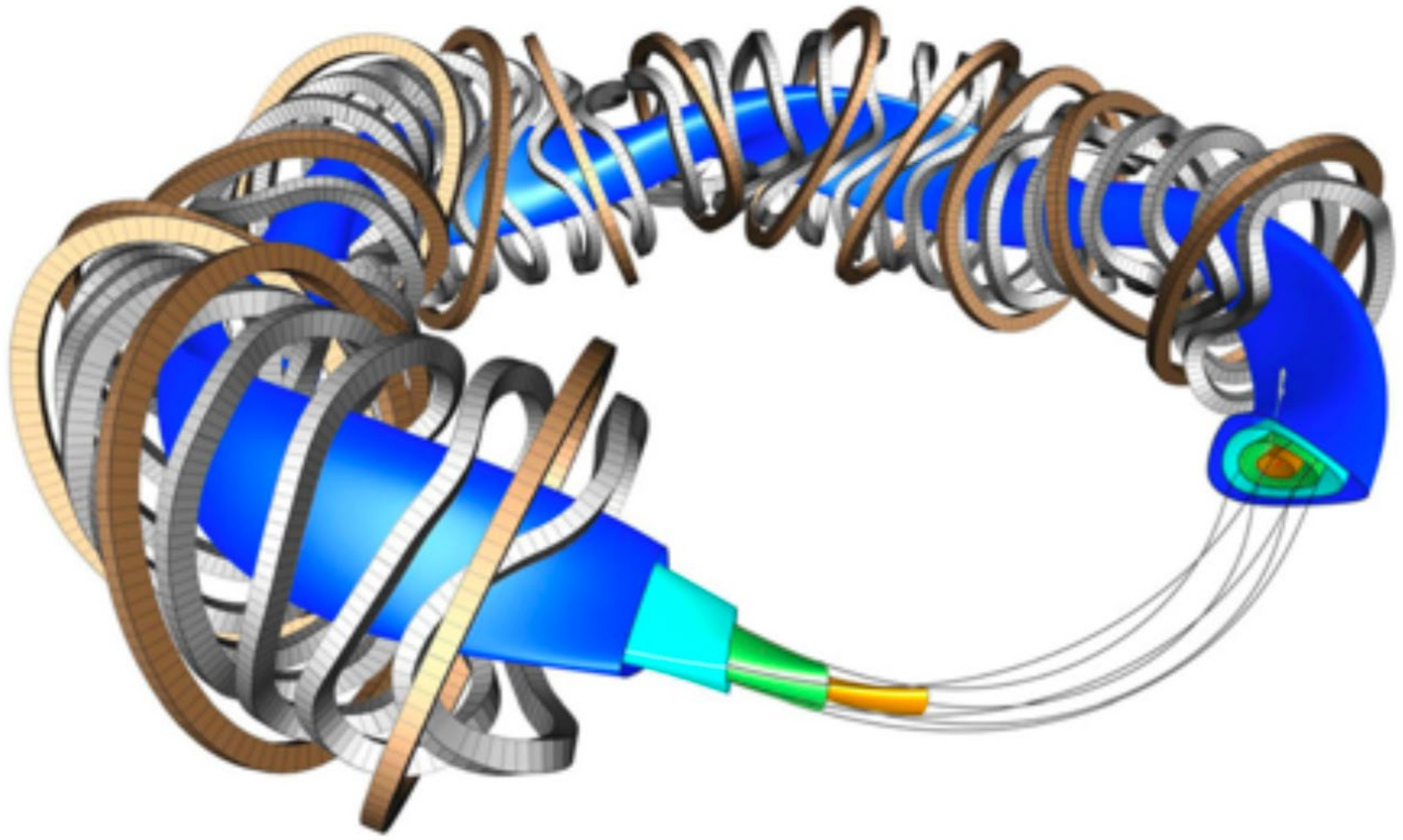
Example: Stellarator equilibria

NCSX: has a discrete three-fold toroidal symmetry.





Wendelstein 7-X: has a discrete five-fold toroidal symmetry.

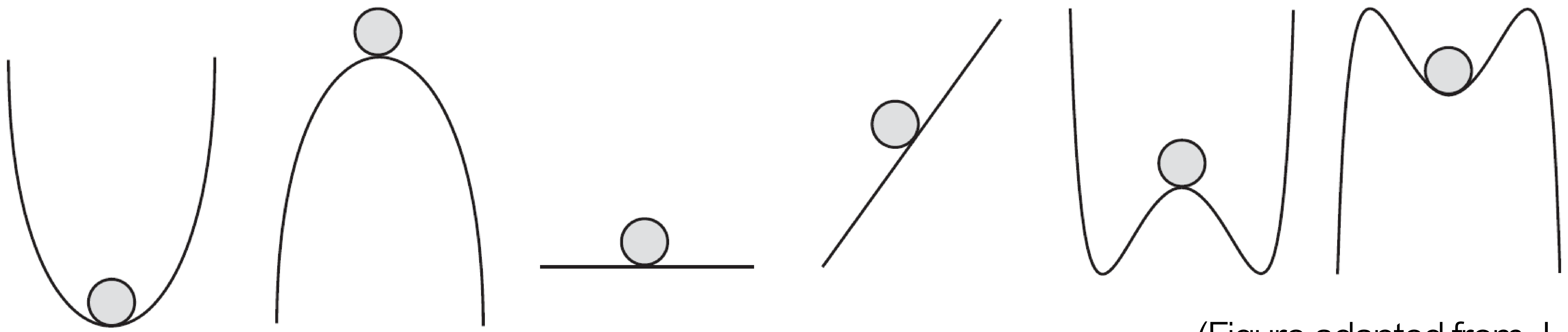


Figures from L.-M. Imbert-Gerard et al., arXiv:1908.05360 (2020).

Linear ideal MHD stability

What is stability?

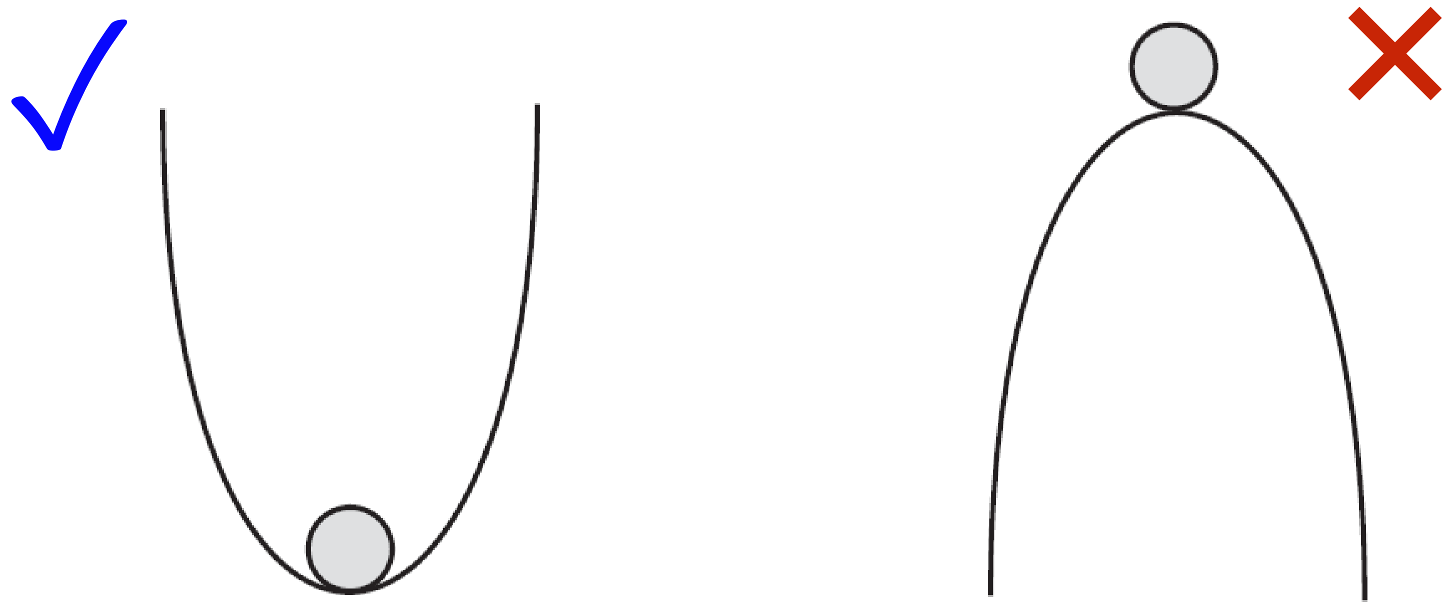
- Recall that steady-state operation is important for magnetic confinement fusion.
- In the MHD regime, we are often interested in states where the plasma is not changing significantly on the time scale of interest.
- Once we have found an equilibrium, we usually want to know whether it is stable or unstable.
- Stability is a concept from dynamical systems theory that tells us what happens to a state when it is perturbed.
- An unstable state moves far away from equilibrium when perturbed. A stable state remains close.
- There are many different types of stability:



(Figure adapted from J. P. Freidberg, Ideal MHD, CUP (2014).)

Linear ideal MHD stability

- For steady-state operation, linearly stable equilibria are typically desirable:



- Analysing linear is stability is comparatively simple since it is local to the equilibrium point.

- Using a perturbation series to expand the ideal MHD evolution equations and retaining only first-order (linear) terms produces the linearised MHD equations.

- Linear ideal MHD stability reduces to solving a linear eigenvalue problem.

Complementary approaches to linear MHD stability

- The eigenvalue problem for linear ideal MHD stability can be solved numerically using a wide variety of eigenvalue codes.

- The linearised MHD evolution equations can also be solved directly.

- Examining special cases using analytic techniques (e.g., perturbation theory, boundary layer theory, asymptotics) produces simple criteria which can be used as metrics for device design and scenario development.

MHD in practice

Practical MHD analysis for fusion plasmas

- There are three components to MHD analysis for fusion plasmas:

Equilibrium calculations	Linear stability analysis	Nonlinear dynamical simulations

Cost, Fidelity →

- They are complementary analyses which, together, form the workflow for a wide range of applications:

  ✓ Physics studies and simulations (e.g., understanding experimental observations)

  ✓ Scenario development and studies (e.g., ITER)

  ✓ Device design and optimisation (e.g., fusion pilot plant concepts, next-generation stellarators)

Thank you!

Email: awright@pppl.gov

: @mini_space_dino